

# Mixture functions and their monotonicity

Jana Špírková

*Department of Quantitative Methods  
and Information Systems,  
Faculty of Economics, Matej Bel University,  
Tajovského 10, 975 90 Banská Bystrica, Slovakia  
e-mail: [jana.spirkova@umb.sk](mailto:jana.spirkova@umb.sk)  
phone: +421908901464  
fax: +421484462000*

Gleb Beliakov

*School of Information Technology, Deakin University,  
221 Burwood Hwy, Burwood 3125, Australia  
e-mail: [gleb@deakin.edu.au](mailto:gleb@deakin.edu.au)  
phone: +61 3 925 17475  
fax: +61 3 925 17604*

Humberto Bustince

*Department of Automation and Computing, Public University of Navarra;  
Institute of Smart Cities, Public University of Navarra,  
Campus Arrosadia, s/n, 31.006 Pamplona, Spain  
e-mail: [bustince@unavarra.es](mailto:bustince@unavarra.es)  
phone: +34 948 16 9254  
fax: +34 948 169169*

Javier Fernandez

*Department of Automation and Computing, Public University of Navarra;  
Institute of Smart Cities, Public University of Navarra,  
Campus Arrosadia, s/n, 31.006 Pamplona, Spain  
e-mail: [fcojavier.fernandez@unavarra.es](mailto:fcojavier.fernandez@unavarra.es)  
phone: +34 948 16 9268  
fax: +34 948 169169*

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## Abstract

We consider mixture functions, which are a type of weighted averages for which the corresponding weights are calculated by means of appropriate continuous functions of their inputs. In general, these mixture function need not be monotone increasing. For this reason we study sufficient conditions to ensure standard, weak and directional monotonicity for specific types of weighting functions. We also analyze directional monotonicity when differentiability is assumed.

*Keywords:* Aggregation function, Mixture function, Monotonicity, Weak monotonicity, Directional monotonicity

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## 1. Introduction

A mixture function is a particular type of weighted averaging operator. To build it, the weights are defined by means of a monotone continuous weighting function and depend on the considered inputs. In this way, it is possible to use the weighting function to give more or less importance to some specific inputs, so mixture functions provide a higher degree of flexibility than usual weighted means, for instance. In this sense, mixture functions can be considered as related to the well-known ordered weighted averaging (OWA) functions [36], but contrary to the latter case, in the former the weights are not assigned a priori but calculated in an input-dependant way. There also exists a close relation between mixture functions and other aggregation functions, such as overlap functions [10], as well as with well-known concepts, as the ROC index [8]. Furthermore, mixture functions can be used in a broad number of applied problems, in fields such as multicriteria decision making, fuzzy systems or data analytics, among others, see [23], [37], [40]. *Note that, since mixture functions extend particular instances of aggregation functions as weighted means, for instance, they can be successfully applied on those problems where the latter are useful. This is specially the case in problems where a reduction of data is required (see, for instance [30], for an application in image processing), and, in general, in any application in machine learning where data fusion plays a relevant role, see [22].*

Recall that a key property in order to define aggregation functions is that of monotonicity [6]. For this reason, different authors have analyzed the problem of whether monotonicity is fulfilled by mixture functions, see [6], [26], and [31], [27]. In particular, in [28], [29] and [32], sufficient conditions to ensure that a mixture function is monotone increasing have been provided.

But usual monotonicity can be a very restrictive condition for applications, and, in fact, some functions which are widely used for data processing, such as the mode function or some kinds of means [6] are not monotone. Some authors have considered the problem of relaxing the monotonicity condition, leading, in particular, to the notion of weak monotonicity [2], [5], [38] and [39]. Basically, a function is weakly monotone if it is monotone along the ray defined by the vector  $(1, \dots, 1)$ , specially to calculate representative values of clusters of data when outliers exist, see [37]. If monotonicity is required along a ray defined by an arbitrary non-null vector, we get the notion of directional monotonicity [11]. These notions have been further extended, considering concepts such as cone monotonicity, monotonicity with respect to coalitions of inputs [4], as well as those of pre-aggregation function ([24]) or ordered directional monotonicity, see [9], [13] and [15] for more details.

Equally important are the papers related to the so-called generalized mixture functions which generalize mixture functions and extend, along with mixture functions, under certain conditions, an important class of aggregation functions, [20]. The authors in [19] studied also directional and ordered directional monotonicity of the generalized mixture functions, and determined some criteria for obtaining generalized mixture functions and so-called bounded generalized mixture functions. Applications of the mentioned generalized mixture functions in machine learning and classification can be found, for example, in [17] and [21].

In this work, we study sufficient conditions to guarantee standard, directional and weak monotonicity of mixture functions with some specific weighting functions. In particular, we consider weighting functions which are given in terms of linear and exponential

functions, as well as by means of linear splines. Furthermore, we also analyze the problem of directional monotonicity for differentiable mixture functions.

The paper consists of seven sections and the Appendix. Section 1 presents an overview of the latest results on monotonicity of mixture functions. Section 2 presents the main definitions. Because the paper introduces also sufficient conditions of standard and weak monotonicity of mixture functions with linear spline weighting function, this section gives presents concepts related to linear spline functions. Section 3 provides sufficient conditions of standard and weak monotonicity of mixture functions with linear and exponential weighting functions. Section 4 introduces sufficient conditions of standard and weak monotonicity of mixture functions with linear spline weighting functions. Section 5 gives sufficient conditions of directional monotonicity of mixture functions with linear and exponential weighting functions. Moreover, it also gives sufficient conditions of ordered directional monotonicity. Section 6 introduces sufficient conditions of directional monotonicity of mixture functions with differentiable weighting functions. The Conclusion summarizes the results and provides some ideas for future research. The Appendix contains proofs of selected theorems. All calculations were made using the R software, [35].

## 2. Preliminaries

Throughout the paper, the following notations are used. We denote by  $\mathbb{I} = [a, b] \subset \overline{\mathbb{R}} = [-\infty, \infty]$  a closed interval. In this way,  $\mathbb{I}^n = \{\mathbf{x} = (x_1, \dots, x_n) \mid x_i \in \mathbb{I}, i = 1, \dots, n\}$  is the set of all vectors  $\mathbf{x}$  whose components lie in the interval  $\mathbb{I}$ . Considering  $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ ,  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n)$ , we say that  $\mathbf{x} \leq \mathbf{y}$  if and only if  $x_i \leq y_i$  for each  $i = 1, \dots, n$ . By increasing we do not forcibly mean strictly increasing.

**Definition 2.1.** *A function  $A : \mathbb{I}^n \rightarrow \mathbb{I}$  is an aggregation function if it is monotone increasing in each variable and satisfies the boundary conditions  $A(\mathbf{a}) = a$ ,  $A(\mathbf{b}) = b$ , where  $\mathbf{a} = (a, a, \dots, a)$ ,  $\mathbf{b} = (b, b, \dots, b)$ .*

**Definition 2.2.** *A function  $M_g : \mathbb{I}^n \rightarrow \mathbb{I}$  given by*

$$M_g(x_1, \dots, x_n) = \frac{\sum_{i=1}^n g(x_i) \cdot x_i}{\sum_{i=1}^n g(x_i)}, \quad (1)$$

where  $g : \mathbb{I} \rightarrow ]0, \infty[$  is a continuous weighting function, is called a mixture function.

**Example 2.3.** *For a simple illustration, let the weighting function  $g : [0, 1] \rightarrow ]0, \infty[$  be given by  $g(x) = 1 + 3x$ . Then the mixture function (1) for  $n = 2$  is  $M_g(x_1, x_2) = \frac{3x_1^2 + 3x_2^2 + x_1 + x_2}{3x_1 + 3x_2 + 2}$ . Moreover, we notice that  $M_g(0, 1) = \frac{4}{5} = 0.8$  and  $M_g(0.1, 1) = \frac{4.13}{5.3} \doteq 0.7792$  and thus  $M_g(0, 1) > M_g(0.1, 1)$ , i.e.,  $M_g$  is not monotone increasing w.r.t. Definition 2.4, and hence it is not an aggregation function.*

### 2.1. Monotonicity

We recall now various types of monotonicity which are at the core of this paper.

**Definition 2.4.** A function  $A : \mathbb{I}^n \rightarrow \mathbb{I}$  is *monotone increasing* if for all  $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ , such that  $\mathbf{x} \leq \mathbf{y}$ , it holds that  $A(\mathbf{x}) \leq A(\mathbf{y})$ .

**Definition 2.5.** [39] A function  $A : \mathbb{I}^n \rightarrow \mathbb{I}$  is *weakly monotone increasing* if  $A(\mathbf{x} + k\mathbf{1}) \geq A(\mathbf{x})$  for all  $\mathbf{x}$  and for any  $k > 0$ ,  $\mathbf{1} = \underbrace{(1, 1, \dots, 1)}_{n \text{ times}}$ , such that  $\mathbf{x}, \mathbf{x} + k\mathbf{1} \in \mathbb{I}^n$ .

Clearly, a monotone increasing function is, in particular, weakly monotone increasing, but the converse may not hold.

The notion of weak monotonicity can be generalized considering the idea of directional monotonicity.

**Definition 2.6.** [11], [14] Let  $\mathbf{r}$  be a real  $n$ -dimensional vector,  $\mathbf{r} \neq \mathbf{0}$ . A function  $A : \mathbb{I}^n \rightarrow \mathbb{I}$  is  *$\mathbf{r}$ -increasing* if for all  $\mathbf{x} \in \mathbb{I}^n$  and all  $k > 0$  such that  $\mathbf{x} + k\mathbf{r} \in \mathbb{I}^n$ , it holds that  $A(\mathbf{x} + k\mathbf{r}) \geq A(\mathbf{x})$ .

**Definition 2.7.** [13], [15] Let  $\mathbf{r} = (r_1, r_2, \dots, r_n)$  be a real  $n$ -dimensional vector,  $\mathbf{r} \neq \mathbf{0}$ . A function  $A : \mathbb{I}^n \rightarrow \mathbb{I}$ ,  $\mathbb{I} = [a, b]$ , is  *$\mathbf{r}$ -ordered increasing* if for all  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{I}^n$  and for any permutation  $\sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  with  $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$  and any  $k > 0$  such that  $b \geq x_{\sigma(1)} + kr_1 \geq \dots \geq x_{\sigma(n)} + kr_n \geq a \in \mathbb{I}^n$  we have  $F(\mathbf{x} + k\mathbf{r}_{\sigma^{-1}}) \geq F(\mathbf{x})$ ;  $\mathbf{r}_{\sigma^{-1}} = (r_{\sigma^{-1}(1)}, \dots, r_{\sigma^{-1}(n)})$ .

Related to the notion of weak monotonicity it appears that of shift-invariance, that we recall now.

**Definition 2.8.** A function  $A : \mathbb{I}^n \rightarrow \mathbb{I}$  is *shift-invariant* if  $A(\mathbf{x} + k\mathbf{1}) = A(\mathbf{x}) + k$ ,  $k > 0$ , whenever  $\mathbf{x}, \mathbf{x} + k\mathbf{1} \in \mathbb{I}^n$  and  $A(\mathbf{x}) + k \in \mathbb{I}$ .

Note that a shift-invariant function  $A$  is weakly monotone increasing, [39].

In many settings, rather than using arbitrary real vectors to deal with directional monotonicity, only vectors from the positive octant  $\mathbb{R}_+^n$  are considered, where  $\mathbb{R}_+^n = \{\mathbf{z} \in \mathbb{R}_+^n | z_i \geq 0, i = 1, 2, \dots, n\}$ .

Recall that vectors  $\mathbf{r} \neq \mathbf{0}$  are called *directions*. Weakly monotone increasing functions are  *$\mathbf{r}$ -increasing* in the direction of vector  $\mathbf{r} = (1, 1, \dots, 1)$ .

It is worth to mention that, from the idea of directional monotonicity, the concept of pre-aggregation function has recently arisen, being a function with the same boundary conditions as an aggregation function but which is directionally monotone instead of monotone, see [24].

Now, we give the concept of cone monotonicity as introduced in [4]. We assume a cone  $\mathcal{C}$  based at the origin and lying in the positive octant  $\mathbb{R}_+^n$ .

**Definition 2.9.** [4] Let  $C \subseteq \mathbb{R}_+^n$  be a non-empty cone  $C = \{\mathbf{x} \in \mathbb{R}_+^n | \mathbf{x} \in C \implies \alpha\mathbf{x} \in C, \forall \alpha \geq 0\}$ . A function  $A : \mathbb{I}^n \rightarrow \mathbb{I}$  is called *cone monotone* with respect to  $C$  if  $A$  is  *$\mathbf{r}$ -increasing* in any direction  $\mathbf{r} \in C$ . The set of functions cone monotone with respect to  $C$  will be denoted  $Mon_C$ .



**Proposition 2.10.** [4], [14] Let  $A : \mathbb{I}^n \rightarrow \mathbb{I}$  be a function,  $\mathbf{x} \in \mathbb{I}^n$ ,  $\mathbf{u}, \mathbf{v}$  be real  $n$ -dimensional non-zero vectors and  $c, h, k$  be all positive constants such that  $\mathbf{x} + c\mathbf{w}$ ,  $\mathbf{x} + ck\mathbf{u}$ ,  $\mathbf{x} + ch\mathbf{v}$  and  $\mathbf{x} + ck\mathbf{u} + ch\mathbf{v} \in \mathbb{I}^n$  with  $\mathbf{w} = k\mathbf{u} + h\mathbf{v}$ . Then, if  $A$  is both  $\mathbf{u}$ -increasing and  $\mathbf{v}$ -increasing, it is also  $\mathbf{w}$ -increasing.

*Proof* With respect to Definition 2.6, we can state

$$A(\mathbf{x} + c\mathbf{w}) = A(\mathbf{x} + ck\mathbf{u} + ch\mathbf{v}) \geq A(\mathbf{x} + ck\mathbf{u}) \geq A(\mathbf{x})$$

or

$$A(\mathbf{x} + c\mathbf{w}) = A(\mathbf{x} + ck\mathbf{u} + ch\mathbf{v}) \geq A(\mathbf{x} + ch\mathbf{v}) \geq A(\mathbf{x}).$$

□

**Corollary 1.** If the function  $A$  is standard monotone increasing then it is  $\mathbf{r}$ -increasing in the positive octant.

Note that a function  $A : \mathbb{I}^n \rightarrow \mathbb{I}$  is monotone if and only if it is  $\mathbf{e}_i$ -directional increasing for each  $i = 1, 2, \dots, n$ , where  $\mathbf{e}_i$  denotes the vector whose  $i$ th coordinate is equal to 1, and all the other coordinates are 0. Besides, if  $A$  is  $\mathbf{e}_i$ -increasing for all  $i = 1, 2, \dots, n$ , then it is cone monotone with respect to positive octant, or  $\mathbf{r}$ -increasing with respect to all vectors in the positive octant.

## 2.2. Sufficient conditions of monotonicity

### 2.2.1. Standard monotonicity

For  $\mathbb{I} = [0, 1]$ , Ribeiro and Marques Pereira in [31] have shown that any increasing differentiable weighting function  $g : [0, 1] \rightarrow ]0, \infty[$  such that

$$g \geq g' \tag{2}$$

yields an increasing mixture function (1).

In [32] an in-depth study of the problem of monotonicity for mixture functions was carried out. In particular, sufficient conditions for monotonicity were provided, more general than (2). We recall here the most relevant results from [32].

**Theorem 2.11.** Mixture function  $M_g : \mathbb{I}^n \rightarrow \mathbb{I}$ ,  $\mathbb{I} = [0, 1]$ , given by (1), is monotone increasing if at least one from the following conditions is satisfied:

- for an increasing, piecewise differentiable weighting function  $g : \mathbb{I} \rightarrow ]0, \infty[$ :

$$g(x) \geq g'(x), \tag{3}$$

$$g(x) \geq g'(x) \cdot (1 - x), \tag{4}$$

for a fixed  $n$ ,  $n > 1$ ,

$$\frac{g^2(x)}{(n-1)g(1)} + g(x) \geq g'(x) \cdot (1 - x). \tag{5}$$

- for a decreasing piecewise differentiable weighting function  $g : \mathbb{I} \rightarrow ]0, \infty[$ :

$$g(x) + g'(x) \geq 0, \quad (6)$$

$$g(x) + g'(x) \cdot x \geq 0, \quad (7)$$

for a fixed  $n, n > 1$ ,

$$\frac{g^2(x)}{(n-1) \cdot g(0)} + g(x) + g'(x) \cdot x \geq 0. \quad (8)$$

See [32] for more sufficient monotonicity conditions for mixture functions and their generalizations.

### 2.2.2. Weak monotonicity of mixture functions

The problem of weak monotonicity for mixture functions and their generalizations was considered in [5], [7], [33] and [34]. In particular, a sufficient condition for a mixture function to be weakly monotone is the following.

**Theorem 2.12.** *Let  $M_g : \mathbb{I}^n \rightarrow \mathbb{I}$ , be a mixture function (1) with differentiable weighting function  $g : \mathbb{I} \rightarrow ]0, \infty[$ . Then  $M_g$  is weakly monotone increasing if*

$$\left( \sum_{i=1}^n g(x_i) \right)^2 + \left( \sum_{i=1}^n g(x_i) \right) \cdot \left( \sum_{i=1}^n x_i \cdot g'(x_i) \right) - \left( \sum_{i=1}^n x_i \cdot g(x_i) \right) \cdot \sum_{i=1}^n g'(x_i) \geq 0 \quad (9)$$

for all  $\mathbf{x} \in \mathbb{I}^n$ .

*Proof*  $M_g$  is directionally differentiable in its domain and hence its weak monotonicity is based on non-negativity of the directional derivative, i.e.,  $\nabla M_g \geq 0$ . Without loss of generality, we can rewrite formula (1) as follows:

$$M_g(\mathbf{x}) = \frac{g(x_j) \cdot x_j + \sum_{i \neq j}^n g(x_i) \cdot x_i}{g(x_j) + \sum_{i \neq j}^n g(x_i)}, \quad (10)$$

whence

$$\frac{\partial M_g}{\partial x_j} = \frac{\left( g(x_j) + g'(x_j) \cdot x_j \right) \cdot \left( \sum_{i=1}^n g(x_i) \right) - \left( \sum_{i=1}^n g(x_i) \cdot x_i \right) \cdot g'(x_1)}{\left( \sum_{i=1}^n g(x_i) \right)^2} \quad (11)$$

It is obvious that by this way it is possible to write partial derivatives with respect to all input values. If we take all their numerators, we immediately get condition (9).  $\square$

**Remark 1.** *On the basis of (10) and (11) standard monotonicity conditions of the mixture function have been determined too. However, in this case we have to assume (standard) monotonicity of the weighting function. Therefore we mention monotonicity conditions separately for mixture function with increasing and with decreasing weighting function, respectively, see Theorem 2.11.*

*But regarding to determination of the weak monotonicity of the mixture function, formula (9) can be used with an application to increasing and also decreasing weighting function.*

Because we discuss properties of mixture functions with affine, exponential, but also with T-spline weighting functions, we give here basic information about them.

### 2.3. T-spline function

A spline function (of degree 1) is piecewise linear continuous function, which can be composed of one segment, but also of two or more segments. The abscissae of the points where the linear segments join together (plus the ends of the interval) are called the knots of the spline. The knots may or may not be equidistant, and  $h$  spline segments result in  $h+1$  knots. For a fixed set of knots, the set of all linear splines forms a linear vector space of dimension  $h+1$ . The traditional B-splines form a convenient basis to express splines as linear combinations of the basis functions, which have the property of local support and add to one for all  $x$  between the spline knots. Closely related T-spline functions [1, 3] form an alternative basis in the space of splines to express monotonicity conditions in a more convenient form, as non-negativity of spline coefficients. Because the weighting functions need to be monotone, we will work in the T-spline representation, formally introduced below, and will also call the resulting linear spline a T-spline to stress its representation.

**Definition 2.13.** *Assume the interval  $\mathbb{I}$  is partitioned into  $h$  segments of equal length and define the basis functions*

$$\begin{aligned} T_0(x) &= 1; x \geq 0, \\ T_s(x) &= \begin{cases} \min \{hx - s + 1, 1\}; & x \geq \frac{s-1}{h}, \\ 0; & \text{otherwise} \end{cases} \end{aligned} \quad (12)$$

for  $s = 1, \dots, h$ . Then the function

$$S(x) = a_0 T_0(x) + \sum_{s=1}^h a_s T_s(x), \quad (13)$$

where  $a_0$  and  $a_s$ ,  $s = 1, 2, \dots, h$  are real constants, referred to as spline coefficients, is called a T-spline function of degree 1.

T-spline functions are continuous and consist of individual linear segments  $S_s(x)$  on the intervals  $[0, t[, [t, 2t[, \dots, [(h-1)t, 1]$ , where  $t = \frac{1}{h}$ . If the coefficients  $a_0, \dots, a_h$  are non-negative, then  $S$  is positive and monotone increasing. For better understanding, we give an example of T-spline function which consists of three segments.

**Example 2.14.** Using Definition 2.13, we present a three segment spline by

$$\begin{aligned} T_0(x) &= 1; x \geq 0, \\ T_1(x) &= \begin{cases} \min \{3x, 1\}; & x \geq 0, \\ 0; & \text{otherwise,} \end{cases} \\ T_2(x) &= \begin{cases} \min \{3x - 1, 1\}; & x \geq \frac{1}{3}, \\ 0; & \text{otherwise,} \end{cases} \\ T_3(x) &= \begin{cases} \min \{3x - 2, 1\}; & x \geq \frac{2}{3}, \\ 0; & \text{otherwise.} \end{cases} \end{aligned}$$

The first segment represents the function  $S_1(x) = a_0 + 3a_1x$  on the interval  $[0, \frac{1}{3}[$ , the second one  $S_2(x) = a_0 + a_1 - a_2 + 3a_2x$  on the interval  $[\frac{1}{3}, \frac{2}{3}[$  and the third one  $S_3(x) = a_0 + a_1 + a_2 - 2a_3 + 3a_3x$  on the interval  $[\frac{2}{3}, 1]$ . This function is shown on Figure 1.

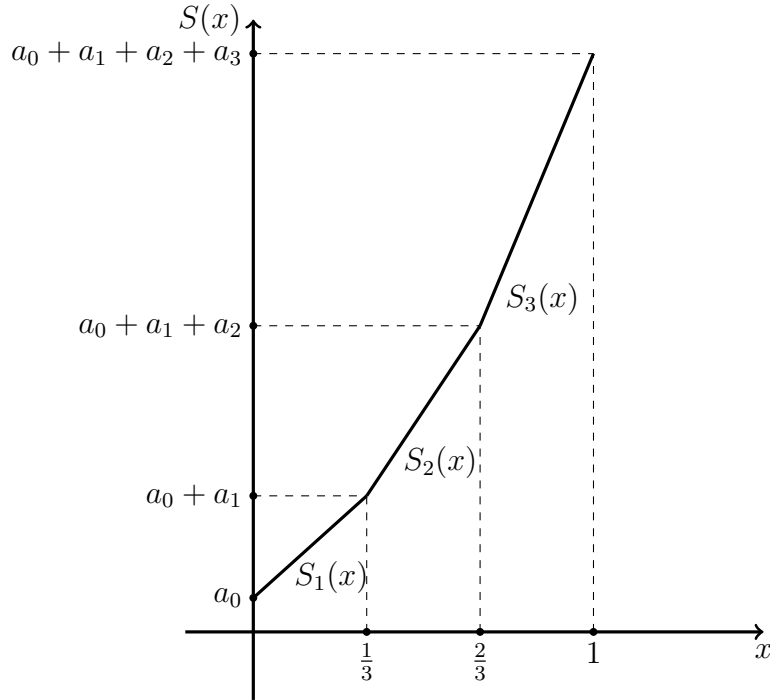


Figure 1: T-spline with three segments.

Obviously, we can construct T-spline functions with more segments in a similar way, as well as consider segments of non-equal length.

### 3. Monotonicity and weak monotonicity of mixture function with affine and exponential weighting function

We discuss sufficient conditions of standard and weak monotonicity of mixture functions with specific types of weighing functions. We start considering mixture functions defined in terms of affine weighting functions  $g(x) = x + l$ ,  $l > 0$ .

#### 3.1. Mixture function with affine weighting function

In the next result we show that we have a high degree of freedom in order to choose the value of the parameter  $l$ .

**Proposition 3.1.** *[33] Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be a mixture function defined by (1), and  $g : [0, 1] \rightarrow ]0, \infty[$  be a weighting function given by  $g(x) = x + l$ ,  $l > 0$ . Then  $M_g$  is:*

1. *for any possible  $n, n > 1$ , standard monotone increasing for*

$$l \geq 1; \tag{14}$$

2. *for a fixed  $n, n > 1$ , standard monotone increasing for*

$$l \geq \sqrt{\frac{n-1}{n}}; \tag{15}$$

3. *for  $n = 2$ , weakly monotone increasing for*

$$l > 0; \tag{16}$$

4. *for a fixed  $n, n \geq 3$ , weakly monotone increasing for*

$$l \geq \frac{n-2}{n}. \tag{17}$$

**Remark 2.** *One can define a mixture function with  $g(0) = 0$  by using limits, if they exist. For example,  $g(x) = x$  produces a well defined mixture function called contraharmonic mean, a special case of the Lehmer mean  $L_m$  with  $m = 1$ . This function is weakly monotone for  $n = 2$ . For more information, see [7].*

**Remark 3.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be a mixture function defined by (1). If  $M_g$  is monotone increasing (weakly monotone increasing), then  $M_{Bg}$ , with  $B > 0$  is also monotone increasing (weakly monotone increasing). Also  $M_{g+B}$  is monotone increasing (weakly monotone increasing).*

**Corollary 2.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the weighting function  $g(x) = cx + 1 - c$ ,  $c \in [0, 1]$ . Then  $M_g$  is:*

1. *for any possible  $n, n > 1$ , standard monotone increasing for*

$$c \in \left[0, \frac{1}{2}\right], \tag{18}$$

2. for a fixed  $n$ ,  $n > 1$ , standard monotone increasing for

$$c \in \left[0, n - \sqrt{n^2 - n}\right], \quad (19)$$

3. for a fixed  $n$ ,  $n > 1$ , weakly monotone increasing for

$$c \in \left[0, \frac{n}{2n-2}\right]. \quad (20)$$

*Proof* On the basis of Remark 3, we can divide the weighting function  $g(x) = cx + 1 - c$  by  $c$ ,  $c \in ]0, 1]$ . The conditions (18), (19) and (20) result from (14), (15) and (17) using substitution  $l = \frac{1}{c} - 1$ . Obviously, for  $c = 0$ ,  $M_g$  is the arithmetic mean which is (standard) monotone and hence weakly monotone.  $\square$

The next theorem provides a global bound on the coefficient  $l$  in the weighting function  $g(x) = x + l$  so that the resulting mixture function is weakly monotone.

### Example 3.2.

With respect to Corollary 2, let function  $g(x) = 0.8x + 0.2$  be the weighting function of the mixture function (1). Then  $M_g(x, y) = \frac{0.8x^2 + 0.8y^2 + 0.2x + 0.2y}{0.8x + 0.8y + 0.4}$  and for input vectors  $(0, 1)$  and  $(0.1, 1)$  the function values are sequentially

$$M_g(0, 1) = \frac{0.8 \cdot 0 + 0.8 \cdot 1 + 0.2 \cdot 0 + 0.2 \cdot 1}{0.8 \cdot 0 + 0.8 \cdot 1 + 0.4} = \frac{1}{1.2} = 0.8333$$

and

$$M_g(0.1, 1) = \frac{0.8 \cdot 0.1^2 + 0.8 \cdot 1 + 0.2 \cdot 0.1^2 + 0.2 \cdot 1}{0.8 \cdot 0.1^2 + 0.8 \cdot 1 + 0.4} = \frac{1.028}{1.28} = 0.8031.$$

Because, we have chosen coefficients of the weighting function outside the interval  $[0, \frac{1}{2}]$  w.r.t (18), and also outside of the interval  $[0, 2 - \sqrt{2}]$  w.r.t (19), we see that monotonicity can be violated. This situation is illustrated also on Figure 2.

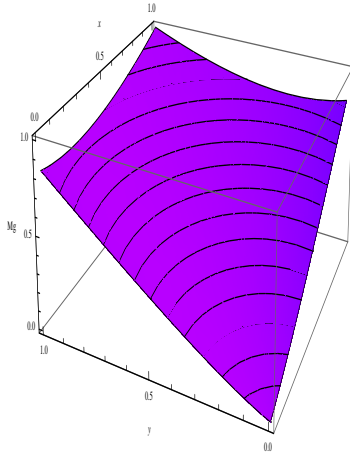


Figure 2:  $M_g$  (1) with  $g(x) = 0.8x + 0.2$ .

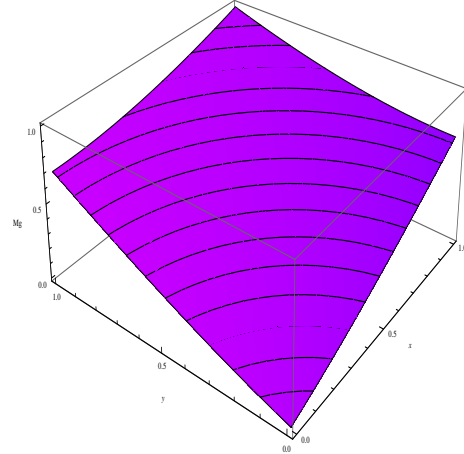


Figure 3:  $M_g$  (1), with  $g(x) = (2 - \sqrt{2})x + \sqrt{2} - 1$ .

Similarly, we can apply weighting function with limiting coefficient  $g(x) = 0.5x + 0.5$  and

hence  $M_g(0, 1) = 0.6667$  and  $M_g(0.1, 1) = 0.6806$ . With respect to (19), we can apply the weighting function  $g(x) = (2 - \sqrt{2})x + \sqrt{2} - 1$ . In this case we have  $M_g(0, 1) = 0.7071$  and  $M_g(0.1, 1) = 0.7111$ . However, monotonicity is maintained for these two input vectors, but, in general, monotonicity of  $M_g$  is also maintained for all input vectors what is obvious from Figures 3, 4 and 5.

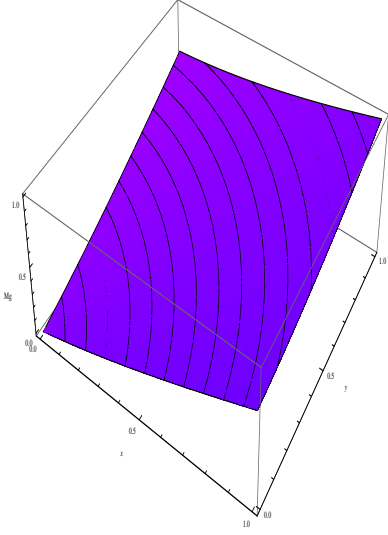


Figure 4:  $M_g(1)$ , with  $g(x) = 0.5x + 0.5$ .

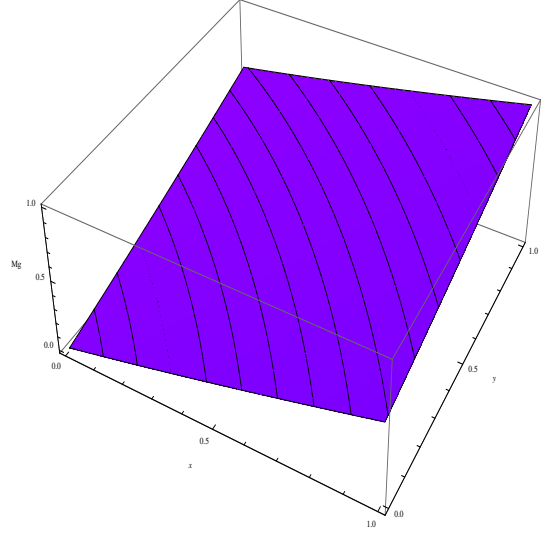


Figure 5:  $M_g(1)$ , with  $g(x) = 0.2x + 0.8$ .

In Theorem 3.3 we give the condition of weak monotonicity of the mixture functions. The link with this theorem is evident in the statement of Theorem 4.4.

**Theorem 3.3.** [7] Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the affine weighting function  $g(x) = x + l$ ,  $l > 0$ . Then  $M_g$  is weakly monotone increasing for all  $n \geq 2$  for

$$l \geq \frac{\sqrt{2} - 1}{2}. \quad (21)$$

**Remark 4.** The right-hand side of condition (21) represents an upper bound for any fixed  $n \geq 2$ . For more information, see Theorem 7 in [7].

**Corollary 3.** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the weighting function  $g(x) = cx + 1 - c$ ,  $c \in [0, 1]$ . Then  $M_g$  is weakly monotone increasing for

$$c \in [0, 2\sqrt{2} - 2]. \quad (22)$$

*Proof* Using Remark 3, substitution  $l = \frac{1}{c} - 1$  and condition (21), we get immediately interval (22). For  $c = 0$ ,  $M_g$  is the arithmetic mean.  $\square$

### 3.2. Mixture function with exponential weighting function

Next we introduce conditions of standard and weak monotonicity of mixture functions with exponential weighting function of the form  $g(x) = \exp(cx) + a$ , where  $c \geq 0$ ,  $a > -1$ . These results complement those in [5]. For statement of standard monotonicity conditions, we used sufficient conditions (3)-(5).

**Proposition 3.4.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1), and  $g(x) = \exp(cx) + a$ ,  $c \geq 0$ ,  $a > -1$ , be the weighting function. Then  $M_g$  is:*

1. *for any possible  $n, n > 1$ , standard monotone increasing for*

$$a > c - 1; \quad (23)$$

2. *for a fixed  $n, n > 1$ , standard monotone increasing for*

$$a > \frac{\exp(c) - 1 + (n-1)c - n(\exp(c) + 1)}{2n} + \frac{\sqrt{(n-1)(c^2(n-1) + (\exp(c) - 1)^2(n-1) + 2c(\exp(c) - 1)(n+1))}}{2n}, \quad (24)$$

whence

- a) *for  $n = 2$ , standard monotone increasing for*

$$a > \frac{c - 3 - \exp(c) + \sqrt{\exp(2c) + \exp(c)(6c - 2) + c^2 - 6c + 1}}{4}, \quad (25)$$

- b) *for  $n = 3$ , standard monotone increasing for*

$$a > \frac{c - 2 - \exp(c) + \sqrt{\exp(2c) + \exp(c)(4c - 2) + c^2 - 4c + 1}}{3}, \quad (26)$$

- c) *for  $n \rightarrow \infty$ , standard monotone increasing for  $a > c - 1$ .*

*Proof*

1. From condition (3), we get  $\exp(cx) + a \geq c \exp(cx)$ . For  $x \rightarrow 0$  with respect to boundary of coefficients  $a$ , we get immediately  $a > c - 1$ .  
From condition (4), we get  $\exp(cx) + a \geq c \exp(cx) \cdot (1 - x)$ . Again, with respect to boundary of coefficients  $a$  and for  $x \rightarrow 0$ , we obtain  $a > c - 1$ .
2. On the basis of condition (5), we get

$$\frac{(\exp(cx) + a)^2}{(n-1)(\exp(c) + a)} + \exp(cx) + a \geq c(1-x) \cdot \exp(cx),$$

whence

$$(\exp(cx) + a)^2 + (n-1)(\exp(c) + a)(\exp(cx) + a) \geq c \exp(cx)(1-x)(n-1)(\exp(c) + a).$$

After reducing and assuming  $x \rightarrow 0$  and boundary of coefficients  $a$ , we get

$$(a+1)^2 + (n-1)(\exp(c) + a)(a+1-c) \geq 0,$$

whence we obtain condition (24).



For the graphical explanation, see Figure 6.  $\square$

Now, we introduce sufficient conditions for weak monotonicity of the mixture function with the same exponential weighting function as in previous case.

**Proposition 3.5.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1), and  $g(x) = \exp(cx) + a$ ,  $c \geq 0$ ,  $a > -1$ , be the weighting function. Then  $M_g$  is:*

*for a fixed  $n$ ,  $n > 1$ , weakly monotone increasing for*

$$a \geq \frac{-[(n-1+\exp(c)) \cdot (2n-c) + cn \exp(c)]}{2n^2} + \frac{\sqrt{[(n-1+\exp(c)) \cdot (2n-c) + cn \exp(c)]^2 - 4n^2(n-1+\exp(c))^2}}{2n^2}, \quad (27)$$

*whence*

*for  $n = 2$ , weakly monotone increasing for  $a > -1$  and  $c \geq 0$ ,*

*for  $n = 3$ , weakly monotone increasing for  $a \geq -0.950581$  and  $c \geq 0$ ,*

*for  $n = 4$ , weakly monotone increasing for  $a \geq -0.892938$  and  $c \geq 0$ ,*

*for  $n = 100$ , weakly monotone increasing for  $a \geq -0.491754$  and  $c \geq 0$ ,*

*for  $n \rightarrow \infty$ , weakly monotone increasing for  $a \geq 0$  and  $c \geq 0$ .*

*Proof* With respect to (9), for the input vector  $(x_1, x_2, \dots, x_n)$ , we get

$$\left( \sum_{i=1}^n \exp(cx_i) + na \right)^2 + \left( \sum_{i=1}^n \exp(cx_i) + na \right) \cdot c \sum_{i=1}^n x_i \exp(cx_i) - \left( \sum_{i=1}^n x_i (\exp(cx_i) + a) \right) \cdot c \sum_{i=1}^n \exp(cx_i) \geq 0.$$

After simplification, assuming symmetry of the mixture function and the input vector  $(0, 0, \dots, 0, 1)$ , we obtain general condition for a weak monotonicity in the form

$$n^2 a^2 + a[(n-1+\exp(c)) \cdot (2n-c) + cn \exp(c)] + (n-1+\exp(c))^2 \geq 0. \quad (28)$$

By solving the previous inequality together with the boundaries of coefficients  $a > -1$ , we obtain condition (27). Moreover, maximal values of the right-hand side of 27 for individual  $n > 1$  represent boundary value of the coefficient  $a$ . For illustration, see Figure 7.  $\square$

**Remark 5.** *Similarly to Remark 2, we can overcome the restriction  $g > 0$  by using limits. Let us assume weighting functions  $g(x) = \exp(cx) + a$  and  $g_\epsilon(x) = \exp(cx) + a + \epsilon$ , for  $a > -1$ ,  $c \geq 0$  and  $\epsilon > 0$ . Then mixture function (1) written in the shape*

$$M_g(x, y) = \begin{cases} M_g(x, y); & x, y \in ]0, 1]; \\ \lim_{\epsilon \rightarrow 0} M_{g_\epsilon}(x, y); & (x, y) = (0, 0) \end{cases} \quad (29)$$

*satisfies conditions in Propositions 3.4 and 3.5 also with non-sharp inequalities.*

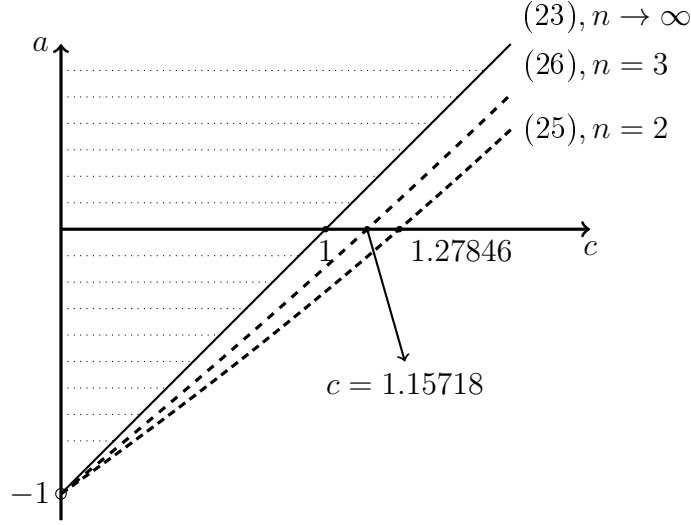


Figure 6: The set of standard monotonicity of  $M_g : [0, 1]^n \rightarrow [0, 1]$  with  $g(x) = \exp(cx) + a$ .

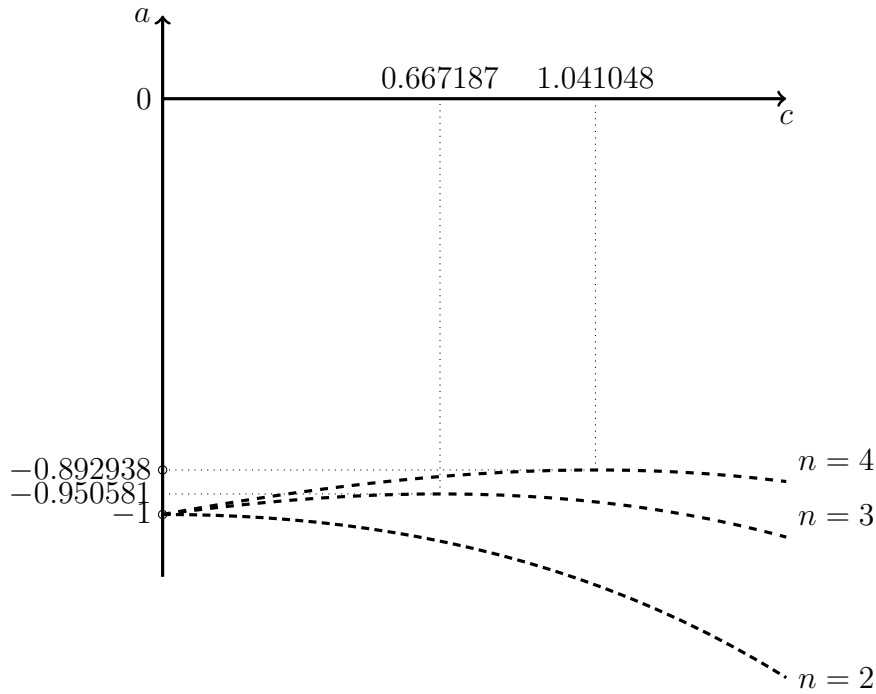


Figure 7: The lower boundary of the set of weak monotonicity of  $M_g : [0, 1]^n \rightarrow [0, 1]$  with  $g(x) = \exp(cx) + a$  for the fixed  $n$ .

#### 4. Monotonicity and weak monotonicity of the mixture function with T-spline weighting function

Now we introduce properties of the mixture function with a piecewise linear weighting function, especially with T-spline weighting function which we described in Subsection 2.3. The main reason for using T-splines is the following. It is known that any continuous function can be approximated arbitrarily well by a piecewise linear function, i.e., by a

linear spline. Hence by using monotone linear splines we can model different weighting functions  $g$ , e.g., study the impact of the shape of the graph of  $g$  on the minimum value of the constant  $a_0 = g(0)$  (if monotonicity holds for  $g(x) = S(x)$  then it also holds for  $g(x) = S(x) + c$ ,  $c > 0$ , hence our interest in the smallest function  $g$  of a particular form). Further, linear splines are piecewise differentiable and hence we can relate the monotonicity conditions to the derivative of  $g$ .

Since the splines are defined through their coefficients  $a_0, a_1, \dots, a_h$ , we will look for conditions which express (weak) monotonicity through inequalities relating these coefficients with spline knots, and in particular a bound on the coefficient  $a_0$ , which can be easily adjusted by translating the graph of the spline up or down. Furthermore, by Remark 3 we have some freedom in selecting another condition on spline coefficients, as they are defined up to an arbitrary positive factor. We shall choose this extra condition in the form  $a_s = 1$ , for some  $s > 0$ , or  $\sum_{s=1}^h a_s = 1$  depending on our needs in the proofs of the following theorems. Therefore the weighting functions which we investigated have the following conditions at the end points of the interval  $[0, 1]$ :  $g(0) = a_0 > 0$  and  $g(1) = a_0 + 1$ .

#### 4.1. Monotonicity of the mixture function with T-spline weighting function

In the next theorem we introduce sufficient condition for a monotone increasingness of the mixture function with the monotone increasing T-spline weighting function.

**Theorem 4.1.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing T-spline weighting function given by Definition 2.13 with  $a_s \geq 0$  and  $\sum_{s=1}^h a_s = 1$ . Then  $M_g$  is monotone increasing for*

$$\min S(x) = a_0 \geq h = \frac{1}{t}. \quad (30)$$

*Proof* In the Appendix. □

In particular, for a spline with  $h$  segments we have the condition  $S \geq S'$ , which is consistent with Theorem 2.12. The derivative of the spline (where it exists – almost everywhere except the knots of the spline, where we can take left or right derivative), is a piecewise constant function which reaches its possible maximum value  $\frac{1}{t} = h$ , and by monotonicity  $\min S(x) = a_0$ . So the two splines for which the condition  $S \geq S'$  is tight are given by  $a_0 = h, a_1 = 0, a_2 = 1$  and  $a_0 = h, a_1 = 1, a_2 = 0$  (in the case  $h = 2$ ), or, more generally  $a_i = 0$  for all  $s = 1, \dots, h$  except one  $a_j = 1$ , and  $a_0 = h$ .

#### 4.2. Weak monotonicity of the mixture function with T-spline weighting function

Now, we introduce the basic result related to monotone increasingness of the mixture function with the monotone increasing T-spline weighting function.

**Theorem 4.2.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing T-spline weighting function defined by Definition 2.13 with  $a_s \geq 0$  and  $\sum_{s=1}^h a_s = 1$ . Then  $M_g$  is weakly monotone increasing for*

$$\min S(x) = a_0 \geq \frac{h}{4} = \frac{1}{4t}. \quad (31)$$

*Proof* In the Appendix. □

Analogously to our reasoning in the previous subsection, we can hypothesise that for a piecewise differentiable continuous function  $g$  the condition of weak monotonicity is  $g \geq \frac{g'}{4}$ . Furthermore, this bound is tight as follows from the proof of Theorem 4.2, i.e., there exists a piecewise differentiable monotone increasing function such that  $g(0) < \frac{g'(0)}{4}$  implies lack of weak monotonicity. This bound can be used to estimate the condition of weak monotonicity for other differentiable weighting functions. For example, take  $g(x) = \ln(x+1) + c$  (cf. [5]). We immediately have  $c \geq \frac{1}{4}$  by using our bound. A detailed analysis of this weighting function with respect to weak monotonicity condition (9) reveals that the actual tight bound on  $c$  is  $c \geq 0.219825$  (for  $n = 5$ ), which is just slightly less than our estimate.

Now, we present condition of weak monotonicity of mixture function with one segment T-spline weighting function to show consistency with the previous results. We recall that the following result corresponds with Theorem 3.3.

**Example 4.3.** *W.r.t. Definition 2.13 let us assume as a weighting function of  $M_g$  T-spline function of degree 1 with two segments. (See also Figure 16 in the Appendix.) We have*

$$\begin{aligned} T_0(x) &= 1; \quad x \geq 0, \\ T_1(x) &= \begin{cases} \min\{2x, 1\}; & x \geq 0, \\ 0; & \text{otherwise,} \end{cases} \\ T_2(x) &= \begin{cases} \min\{2x - 1, 1\}; & x \geq \frac{1}{2}, \\ 0; & \text{otherwise.} \end{cases} \end{aligned}$$

*The first segment represents the function  $S_1(x) = a_0 + 2a_1x$ ,  $a_0 > 0$ , on the interval  $[0, \frac{1}{2}[$  and the second one  $S_2(x) = a_0 + a_1 - a_2 + 2a_2x$  on the interval  $[\frac{1}{2}, 1]$ . For  $a_1 = 1$  and  $a_2 = 0$ , and moreover assuming violation of the boundary condition in Theorem 4.1 we take  $S_1(x) = 0.5 + 2x$  and  $S_2(x) = 1.5$ . These two functions represent T-spline function and we apply them as a weighting function  $g(x)$  of  $M_g$  which is given by (1). That means*

$$g(x) = \begin{cases} 0.5 + 2x; & x \in [0, \frac{1}{2}[, \\ 1.5; & x \in [\frac{1}{2}, 1]. \end{cases}$$

*We compare two function values of  $M_g$ :*

$$M_g(0, 1) = \frac{(0.5 + 2 \cdot 0) \cdot 0 + 1.5 \cdot 1}{0.5 + 2 \cdot 0 + 1.5} = \frac{1.5}{2} = 0.75$$

*and*

$$M_g(0.1, 1) = \frac{(0.5 + 2 \cdot 0.1) \cdot 0.1 + 1.5 \cdot 1}{0.5 + 2 \cdot 0.1 + 1.5} = \frac{1.57}{2.2} = 0.71364,$$

*hence monotonicity is violated. This fact is also shown for whole domain  $[0, 1]^2$  on Figure 8.*

*If we assume coefficients which satisfy the condition given in Theorem 4.1, we obtain increasing function  $M_g$  which is illustrated on Figure 9.*

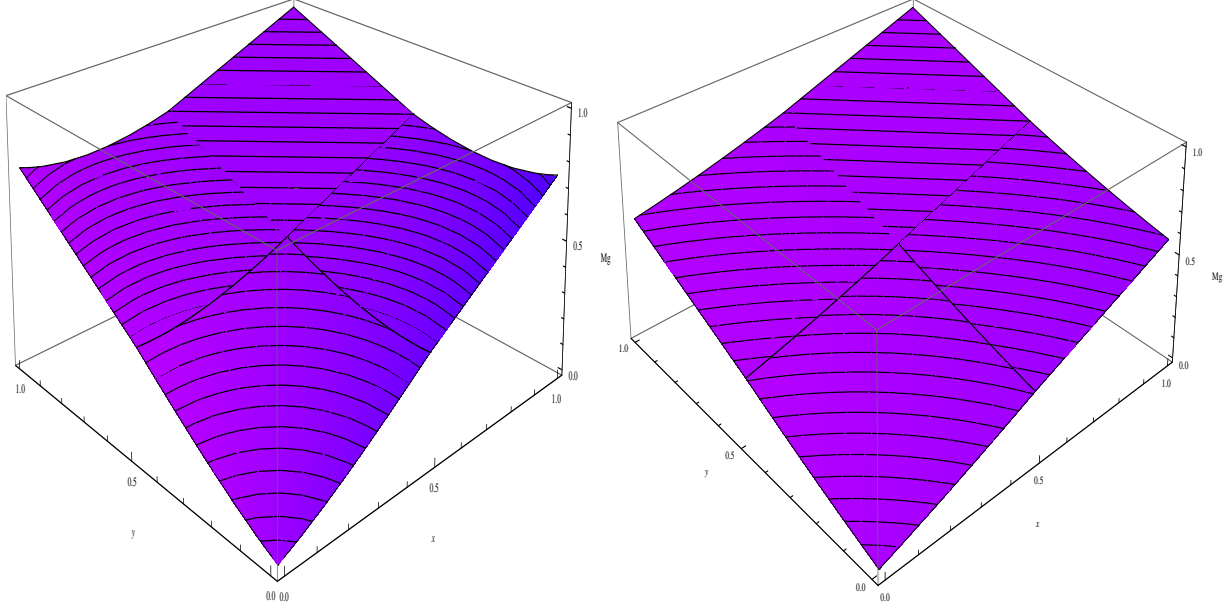


Figure 8:  $M_g$  (1),  $S_1(x) = 0.5 + 2x$ , for  $x \in [0, 0.5[$ ,  $S_2(x) = 1.5$ , for  $x \in [0.5, 1]$  Figure 9:  $M_g$  (1),  $S_1(x) = 3 + 2x$ , for  $x \in [0, 0.5[$ ,  $S_2(x) = 4$ , for  $x \in [0.5, 1]$

**Theorem 4.4.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing T-spline weighting function defined by Definition 2.13 with one segment. Then  $M_g$  is weakly monotone increasing for*

$$a_0 \geq \frac{\sqrt{2} - 1}{2}. \quad (32)$$

*Proof* In the Appendix. □

#### 4.3. Weak monotonicity of the mixture function with two segment T-spline weighting function with a general knot

Now, we give a definition of two segment T-spline function with a general knot  $t \in ]0, 1[$ .

**Definition 4.5.** *Assume the functions*

$$\begin{aligned} {}_tT_0(x) &= 1; \quad x \geq 0, \\ {}_tT_1(x) &= \begin{cases} \min \left\{ \frac{1}{t}x, 1 \right\}; & x \geq 0, \\ 0; & \text{otherwise,} \end{cases} \\ {}_tT_2(x) &= \begin{cases} \min \left\{ \frac{1}{1-t}x - \frac{t}{1-t}, 1 \right\}; & x \geq t, \\ 0; & \text{otherwise,} \end{cases} \end{aligned} \quad (33)$$

where  $t \in ]0, 1[$ . Then the function

$${}_tS(x) = a_0 \cdot {}_tT_0(x) + \sum_{s=1}^2 a_s \cdot {}_tT_s(x), \quad (34)$$

where  $a_0$  and  $a_s$  are constants, is a two segment linear  $T$ -spline function with the knot  $t$  in general position. This  $T$ -spline function is also given by  ${}_tS_1(x) = a_0 + \frac{a_1}{t}x$  on the interval  $[0, t[$  and by  ${}_tS_2(x) = a_0 + a_1 - \frac{a_2 t}{1-t} + \frac{a_2}{1-t}x$  on the interval  $[t, 1]$ .

**Theorem 4.6.** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing  $T$ -spline weighting function defined by Definition 4.5 with  $a_s \geq 0$  and  $\sum_{s=1}^2 a_s = 1$ . Then  $M_g$  is monotone increasing for

$$\min {}_tS(x) = a_0 \geq \frac{1}{t}. \quad (35)$$

*Proof* In the Appendix. □

Note that contrary to Theorem 4.1  $t$  is not fixed at  $\frac{1}{h} = \frac{1}{2}$ .

**Theorem 4.7.** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing  $T$ -spline weighting function defined by Definition 4.5 with  $a_s \geq 0$  and  $\sum_{s=1}^2 a_s = 1$ . Then  $M_g$  is weakly monotone increasing for

$$\min {}_tS(x) = a_0 \geq \frac{1}{4t}. \quad (36)$$

*Proof* In the Appendix. □

We can now provide the following interpretation of the above mentioned results. The worst case scenario (in terms of the lower bound for  $a_0$ ) happens when  $t \rightarrow 0$  or  $t \rightarrow 1$ , in which case the value  $\frac{1}{t}$  approximates the derivative of the spline on one of the segments. Taking into account the normalisation condition  $a_1 + a_2 = 1$  and that  $S \geq S'$ , we obtain

$$\frac{a_0}{a_2} \geq \frac{a_1}{a_2}.$$

## 5. Directional monotonicity of the mixture function with affine and exponential weighting function

We now discuss directional monotonicity for mixture functions which are defined by means of affine weighting functions.

**Theorem 5.1.** Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1) with the affine weighting function  $g(x) = x + l$ ,  $l > 0$ . Then  $M_g$  is  $\mathbf{r}$ -increasing for vectors  $\mathbf{r} = (r_1, r_2)$ ,  $\mathbf{r} \neq \mathbf{0}$ ,  $r_1 + r_2 > 0$  which satisfy the condition

$$l > \max \left\{ \frac{-r_1}{r_1 + r_2}, \frac{-r_2}{r_1 + r_2} \right\} + \sqrt{\frac{r_1^2 + r_2^2}{2(r_1 + r_2)^2}}. \quad (37)$$

*Proof* Let  $\mathbf{r} = (r_1, r_2) \neq \mathbf{0}$ . Let  $\mathbf{x} = (x, y) \in \mathbb{I}^2$  and  $k > 0$  such that  $\mathbf{x} + k\mathbf{r} \in \mathbb{I}^2$ . From Definition 2.6 we get

$$\frac{(x + kr_1)(x + kr_1 + l) + (y + kr_2)(y + kr_2 + l)}{x + y + 2l + k(r_1 + r_2)} \geq \frac{x(x + l) + y(y + l)}{x + y + 2l},$$

whence

$$(x + y + 2l) (2(xr_1 + yr_2) + k(r_1^2 + r_2^2) + l(r_1 + r_2)) \geq (r_1 + r_2)(x^2 + y^2 + l(x + y)).$$

Without loss of generality, for  $k \rightarrow 0$  and after some modification, we obtain inequality

$$(x^2 - y^2)(r_1 - r_2) + 2xy(r_1 + r_2) + 4l(xr_1 + yr_2) + 2l^2(r_1 + r_2) \geq 0. \quad (38)$$

Assume  $r_1 + r_2 > 0$ . The left-hand side of inequality (38) is a quadratic function with non-negative discriminant. We need to set maximum value for  $l$  for it to be non-negative for all input values and corresponding vectors. Therefore we need to maximize the root of the quadratic function

$$l = \frac{-4(xr_1 + yr_2) + \sqrt{16(r_1x + r_2y)^2 - 8(r_1 + r_2) \cdot ((x^2 - y^2)(r_1 - r_2) + 2xy(r_1 + r_2))}}{4(r_1 + r_2)}.$$

It is apparent that it is sufficient to maximize  $-4(xr_1 + yr_2)$ . On the basis of selections  $r_1 > 0, r_2 < 0$  and input  $(0, 1)$ ;  $r_1 < 0, r_2 > 0$  using input  $(1, 0)$ , we get restrictions on the coefficient  $l$  in the form (37).  $\square$

**Corollary 4.** Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1) with the affine weighting function  $g(x) = x + l$ ,  $l > 0$ . Then  $M_g$  is  $\mathbf{r}$ -increasing with respect to condition (38) as follows, see Figure 10.

1. For  $l \rightarrow 0^+$ , the condition  $r_2 = r_1$  must be satisfied.  
Then  $M_g$  is  $\mathbf{r}$ -increasing only in the direction  $(r_1, r_2)$  where  $r_1 = r_2 > 0$ , i.e., it is a proper weakly monotone function;
2. For  $l = 0.5$ , we get the conditions  $0.28r_1 \leq r_2 \leq 3.75r_1$ ;
3. For  $l = \frac{\sqrt{2}}{2}$ , we get the conditions  $r_1 \geq 0$  and  $r_2 \geq 0$  (standard monotonicity);
4. For  $l = 1$ , we get the conditions  $r_2 \geq -7r_1$  and  $r_2 \geq -\frac{1}{7}r_1$ ;
5. For  $l = 2$ , the conditions  $r_2 \geq -\frac{17}{7}r_1$  and  $r_2 \geq -\frac{7}{17}r_1$  must be satisfied.
6. For  $l \rightarrow \infty$ , the condition  $r_2 > -r_1$  must be satisfied.  
Then  $M_g$  is  $\mathbf{r}$ -increasing in all directions which are determined by the half-plane.

*Proof*

- Conditions 1 - 5 are obtained directly from condition (38).
- Conditions 6 can be shown by

$$\lim_{r_2 + r_1 \rightarrow 0^+} \frac{-r_1}{r_1 + r_2} + \sqrt{\frac{r_1^2 + r_2^2}{2(r_1 + r_2)^2}} = \infty, \quad \lim_{r_1 + r_2 \rightarrow 0^+} \frac{-r_2}{r_1 + r_2} + \sqrt{\frac{r_1^2 + r_2^2}{2(r_1 + r_2)^2}} = \infty.$$

$\square$

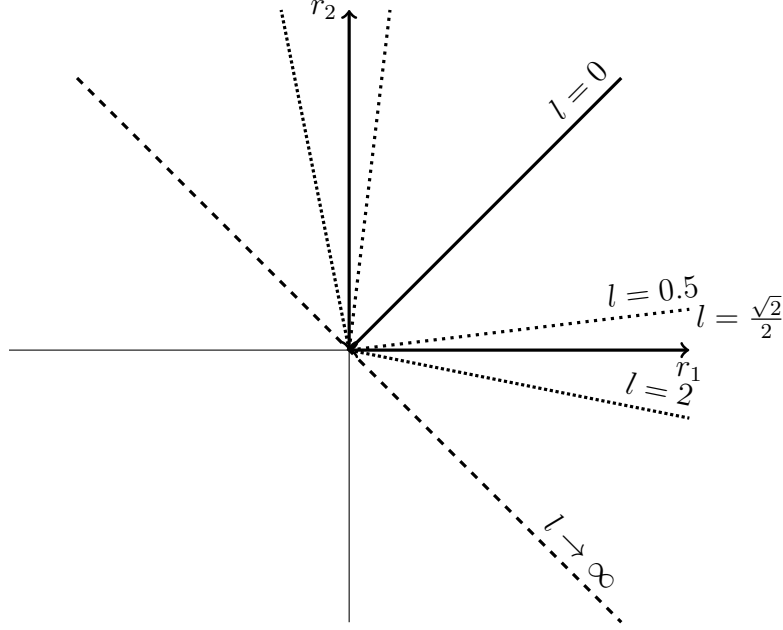


Figure 10: The sets of directional monotonicity of  $M_g : [0, 1]^2 \rightarrow [0, 1]$  with  $g(x) = x + l$ .

**Corollary 5.** Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1) with the affine weighting function  $g(x) = x + l$ ,  $l > 0$ . Then  $M_g$  is  $\mathbf{r}$ -increasing for non-zero vectors  $\mathbf{r} = (r, 1 - r)$ ,  $r \geq 0$  which satisfy the conditions

$$\begin{aligned}
 l &> -r + \sqrt{\left(r - \frac{1}{2}\right)^2 + \frac{1}{4}} \quad \text{for } 0 \leq r \leq \frac{1}{2} \\
 \text{or} \\
 l &> r - 1 + \sqrt{\left(r - \frac{1}{2}\right)^2 + \frac{1}{4}} \quad \text{for } \frac{1}{2} \leq r \leq 1.
 \end{aligned} \tag{39}$$

*Proof* Let  $\mathbf{r} = (r, 1 - r)$ ,  $r \geq 0$ . Let  $\mathbf{x} = (x, y) \in \mathbb{I}^2$  and  $k > 0$  such that  $\mathbf{x} + k\mathbf{r} \in \mathbb{I}^2$ . From Definition 2.6 we get

$$\frac{(x + kr)(x + kr + l) + (y + k(1 - r))(y + k(1 - r) + l)}{x + y + 2l + k} \geq \frac{x(x + l) + y(y + l)}{x + y + 2l}, \tag{40}$$

whence

$$(x + y + 2l) [2(rx + (1 - r)y) + k(r^2 + (1 - r)^2) + l] \geq x^2 + y^2 + xl + yl. \tag{41}$$

Without loss of generality, for  $k \rightarrow 0$  and after some modification, we obtain inequality

$$2l^2 + 4l(r(x - y) + y) + 2xy + (x^2 - y^2)(2r - 1) \geq 0. \tag{42}$$

Using the same consideration as in the proof of Theorem 5.1, we obtain for  $x \rightarrow 0$ ,  $y \rightarrow 1$  or  $x \rightarrow 1$ ,  $y \rightarrow 0$ , conditions (39). See Figure 11.  $\square$



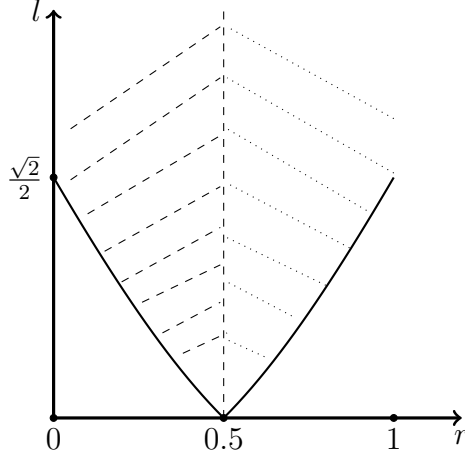


Figure 11: The set of directional monotonicity of  $M_g : [0, 1]^2 \rightarrow [0, 1]$  with  $g(x) = x + l$ .

Regarding the ordered directional monotonicity, mixture function  $M_g : [0, 1]^2 \rightarrow [0, 1]$  with the weighting function  $g(x) = x + l$ ,  $l > 0$ , is symmetric and hence it is ordered  $\mathbf{r}$ -increasing for the same coefficient  $l$  stated in Corollary 5.

**Proposition 5.2.** *Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1) with the affine weighting function  $g(x) = x + l$ ,  $l > 0$  and let  $\mathbf{r} = (r, 1 - r)$ ,  $r \geq 0$ . If  $M_g$  is  $\mathbf{r}$ -ordered increasing, then the coefficient  $l$  must satisfy the condition*

$$l > -r + \sqrt{\left(r - \frac{1}{2}\right)^2 + \frac{1}{4}} \quad \text{for } 0 \leq r \leq \frac{1}{2}. \quad (43)$$

*Proof* Let  $\mathbf{r} = (r, 1 - r)$ ,  $r \geq 0$ . Let  $\mathbf{x} = (x, y) \in [0, 1]^2$  and  $k > 0$  such that  $\mathbf{x} + k\mathbf{r} \in [0, 1]^2$ . Using Definition 2.7, if  $x > y$ , we get gradually (40), (41) and (42). For  $x \rightarrow 1$ ,  $y \rightarrow 0$ , we obtain condition (43).

If  $x < y$ , it is enough to replace  $x$  and  $y$  in inequality (42) and use boundary input vector  $(0, 1)$ , from where we obtain condition (43), again. The set of directions for which  $M_g$  is  $\mathbf{r}$ -ordered increasing is represented by the left part of Figure 11 which is highlighted by dashed lines.  $\square$

**Remark 6.** *From the symmetry of mixture function  $M_g : [0, 1]^2 \rightarrow [0, 1]$ , with the weighting function  $g(x) = x + l$ ,  $l > 0$ , it is obvious that if we consider vector  $\mathbf{r} = (1 - r, r)$ , we obtain the second part of condition (39), i.e.,*

$$l > r - 1 + \sqrt{\left(r - \frac{1}{2}\right)^2 + \frac{1}{4}} \quad \text{for } \frac{1}{2} \leq r \leq 1 \quad (44)$$

and graphically, the right part of Figure 11.

**Remark 7.** *With respect to Remark 2, let us assume weighting function  $g_\epsilon(x) = x + \epsilon$ ,  $\epsilon > 0$ . From Corollary 5 we can state that the Lehmer mean*

$$L_1(x, y) = \begin{cases} \frac{x^2 + y^2}{x + y}; & x, y \in ]0, 1]; \\ \lim_{\epsilon \rightarrow 0} M_{g_\epsilon}(x, y); & (x, y) = (0, 0) \end{cases} \quad (45)$$

is  $\mathbf{r}$ -increasing only for vector  $\mathbf{r} = (\frac{1}{2}, \frac{1}{2})$ , so it is just weakly monotone increasing.

**Remark 8.** Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1). If  $M_g$  is  $\mathbf{r}$ -increasing, then  $M_{Bg}$ , with  $B > 0$  is also  $\mathbf{r}$ -increasing.

The next corollary gives us a sufficient condition for directional monotonicity of mixture function (1) with the weighting function  $g(x) = cx + 1 - c$ ,  $c \in [0, 1]$ .

**Corollary 6.** Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1) with the weighting function  $g(x) = cx + 1 - c$ ,  $c \in [0, 1]$ . Then  $M_g$  is  $\mathbf{r}$ -increasing for vectors  $\mathbf{r} = (r_1, r_2)$  which satisfy the following conditions for  $c$ :

$$\begin{aligned} 0 \leq c &\leq \frac{\sqrt{2}(r_1 + r_2)}{\sqrt{2}r_2 + \sqrt{r_1^2 + r_2^2}} \\ \text{and} \\ 0 \leq c &\leq \frac{\sqrt{2}(r_1 + r_2)}{\sqrt{2}r_1 + \sqrt{r_1^2 + r_2^2}}. \end{aligned} \tag{46}$$

*Proof* Using Remark 8, conditions (37) together with the substitution  $l = \frac{1}{c} - 1$ ,  $c \in ]0, 1]$ , we get condition (46). For  $c \rightarrow 0^+$ ,  $M_g$  becomes the arithmetic mean, which is monotone increasing for all  $\mathbf{r} : r_1 + r_2 \geq 0$ .  $\square$

**Corollary 7.** Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1) with the weighting function  $g(x) = cx + 1 - c$ ,  $c \in [0, 1]$ . Then  $M_g$  is  $\mathbf{r}$ -increasing for vectors  $\mathbf{r} = (r_1, r_2)$  which satisfy conditions

- for  $c \in [0, 2 - \sqrt{2}[$ ,

$$r_2 \geq \frac{c^2 - 4c + 2}{c^2 - 2} r_1 \quad \text{and} \quad r_2 \geq \frac{c^2 - 2}{c^2 - 4c + 2} r_1; \tag{47}$$

- for  $c \in ]2 - \sqrt{2}, 1]$ ,

$$r_2 \geq \frac{c^2 - 4c + 2}{c^2 - 2} r_1 \quad \text{and} \quad r_2 \leq \frac{c^2 - 2}{c^2 - 4c + 2} r_1; \tag{48}$$

- for  $c = 2 - \sqrt{2}$ ,

$$r_1 \geq 0 \quad \text{and} \quad r_2 \geq 0. \tag{49}$$

So this mixture function is directionally monotone increasing in the following cases, for instance:

1. for  $c = 0$  for every direction  $\mathbf{r} \neq \mathbf{0}$  in the half-plane highlighted in Figure 12;
2. for  $c = 2 - \sqrt{2}$  for every direction in the first quadrant;
3. for  $c = 0.7$  for every direction in the highlighted acute angle.

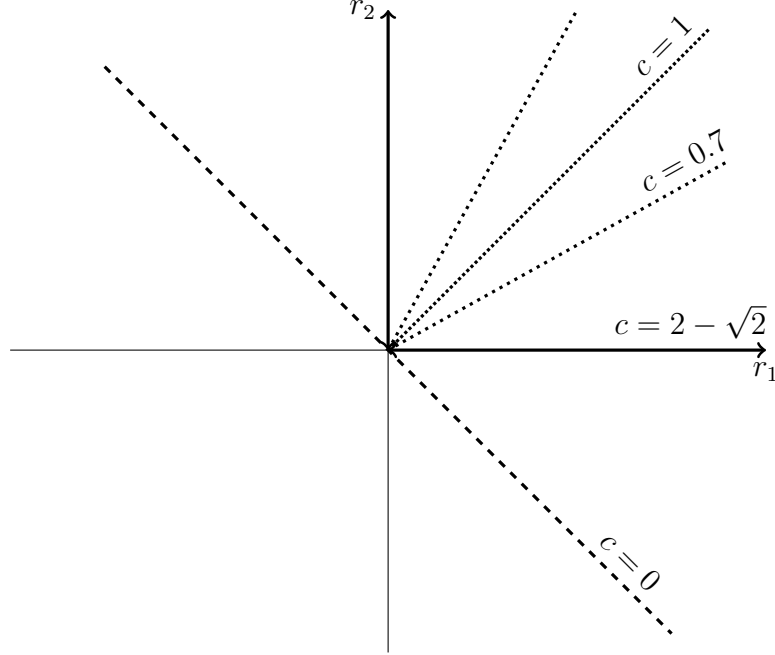


Figure 12: The set of directional monotonicity of  $M_g : [0, 1]^2 \rightarrow [0, 1]$  with  $g(x) = cx + 1 - c$ .

It is clear, both from our proof and from Figure 12, that the set directions for which directionally increasing monotonicity holds becomes gradually smaller from upper half-plane bounded by line  $r_2 = -r_1$  (for  $c = 0$ ) to half line  $r_2 = r_1$  (for  $c = 1$ ).

**Corollary 8.** *Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1) with the weighting function  $g(x) = cx + 1 - c$ ,  $c \in [0, 1]$ . Then  $M_g$  is  $\mathbf{r} = (r, 1 - r)$ -increasing,  $r \geq 0$ ,  $\mathbf{r} \neq \mathbf{0}$  for coefficients  $c$  which satisfy conditions*

$$\begin{aligned}
 0 &\leq c \leq \frac{1 - r - \sqrt{(r - \frac{1}{2})^2 + \frac{1}{4}}}{\frac{1}{2} - r} \quad \text{for } 0 \leq r \leq \frac{1}{2} \\
 \text{or} \\
 0 &\leq c \leq \frac{r - \sqrt{(r - \frac{1}{2})^2 + \frac{1}{4}}}{r - \frac{1}{2}} \quad \text{for } 1 \geq r \geq \frac{1}{2}.
 \end{aligned} \tag{50}$$

*Proof* Using Corollary 5, Remark 3 and substitution  $l = \frac{1}{c} - 1$ , we obtain conditions (50) with the convention  $\frac{0}{0} = 1$ . The situation is illustrated on Figure 13.  $\square$

**Example 5.3.** *For illustration of directional monotonicity of  $M_g$  in a direction  $\mathbf{r} = (r, 1 - r)$  w.r.t. Definition 2.6 and Corollary 8, assume  $M_g$  given by (1) with  $g(x) = cx + 1 - c$  and  $\mathbf{r} = (0.9, 0.1)$ . On the basis of condition (50), the limiting case for  $c$  to maintain directional monotonicity, is  $c \leq 0.649219$ . Therefore, we consider two situations, one for  $c = 0.9$  and the other for  $c = 0.5$ . We compare two function values of  $M_g : [0, 1]^2 \rightarrow [0, 1]$ , where input values are  $\mathbf{x} = (x, y) = (0, 0.9)$ ,  $\mathbf{r} = (0.9, 0.1)$ ,  $k = 0.1$ , and  $g(x) = 0.9x + 0.1$ .*

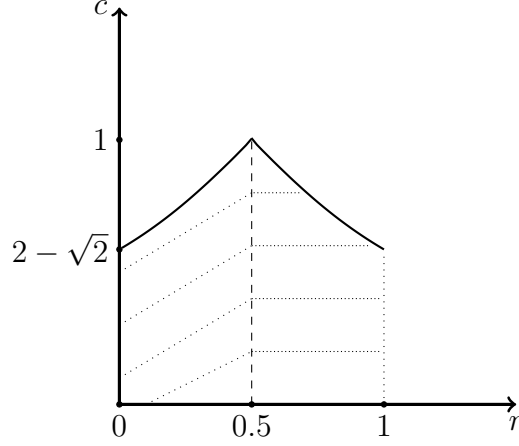


Figure 13: The set of directional monotonicity of  $M_g : [0, 1]^2 \rightarrow [0, 1]$  with  $g(x) = cx + 1 - c$ .

*Then  $M_g(\mathbf{x} + k\mathbf{r}) = 0.775073$  and  $M_g(\mathbf{x}) = 0.810891$ , hence directional monotonicity is violated. If we consider  $M_g$  with  $g(x) = 0.5x + 0.5$ , directional monotonicity is maintained. The visualization of these situations is in the Figures 14 and 15. Although in Figure 15 we considered the situation with  $k = 0.1$ , but the situation is the same for all the permissible  $k$ .*

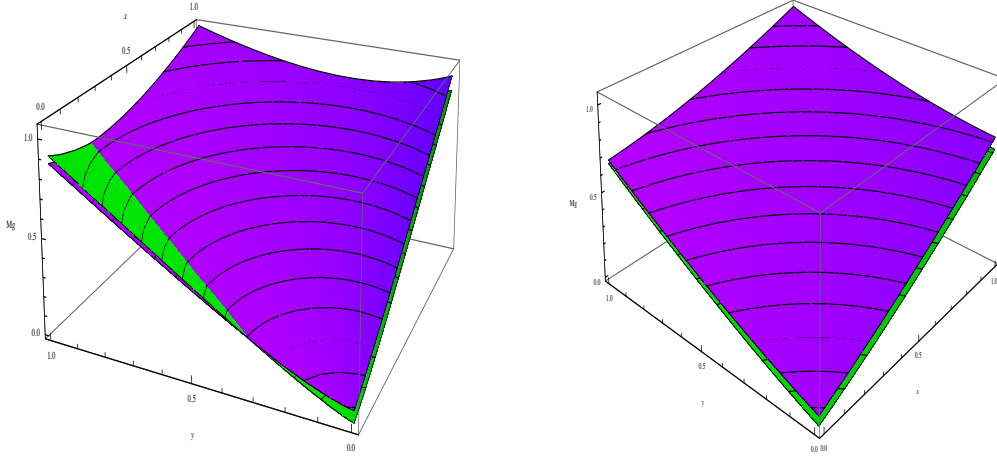


Figure 14:  $M_g(\mathbf{x} + k\mathbf{r})$  and  $M_g(\mathbf{x})$  (1) with  $g(x) = 0.9x + 0.1$ . Figure 15:  $M_g(\mathbf{x} + k\mathbf{r})$  and  $M_g(\mathbf{x})$  (1), with  $g(x) = 0.5x + 0.5$ .

Now we consider directional monotonicity for mixture function defined by means of an exponential weighting function of the form  $g(x) = \exp(cx) + a$ , that we have already commented previously.

**Proposition 5.4.** *Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be the mixture function defined by (1), and  $g(x) = \exp(cx) + a$ ,  $c \geq 0$ ,  $a > -1$ . Let  $\mathbf{r} = (r_1, r_2)$  be a two-dimensional vector with  $r_1, r_2 \geq 0$ . Then  $M_g$  is  $\mathbf{r}$ -increasing on its whole domain.*

*Proof* According to Definition 2.6, we can state

$$\begin{aligned} & \frac{(x + kr_1) \cdot (\exp(c(x + kr_1)) + a) + (y + kr_2) \cdot (\exp(c(y + kr_2)) + a)}{\exp(c(x + kr_1)) + \exp(c(y + kr_2)) + 2a} \geq \\ & \geq \frac{(\exp(cx) + a) \cdot x + (\exp(cy) + a) \cdot y}{\exp(cx) + \exp(cy) + 2a}. \end{aligned} \quad (51)$$

The smallest values of the left-hand side are achieved for the input vector  $(x, y) = (0, 1)$  and, taking symmetry into account, also for the vector  $(x, y) = (1, 0)$ . After standard modification, we get for the input vector  $(0, 1)$

$$\frac{kr_1 \cdot (\exp(ckr_1) + a) + (1 + kr_2) \cdot (\exp(c(1 + kr_2)) + a)}{2a + \exp(ckr_1) + \exp(c(1 + kr_2))} \geq \frac{a + \exp(c)}{1 + 2a + \exp(c)}. \quad (52)$$

For  $k \rightarrow 0$  our condition is satisfied.  $\square$

**Remark 9.** Let  $M_g : [0, 1]^2 \rightarrow [0, 1]$  be a mixture function defined by (1). If  $M_g$  is  $\mathbf{r}$ -ordered increasing, then  $M_{Bg}$ , with  $B > 0$  is also  $\mathbf{r}$ -ordered increasing.

## 6. Directional monotonicity of the mixture function with differentiable weighting function

Now, we introduce sufficient conditions of directional monotonicity of mixture function (1), which are based on the directional derivative of the mixture function.

**Proposition 6.1.** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with a differentiable weighting function  $g : [0, 1] \rightarrow ]0, \infty[$  and  $\mathbf{r} = (r_1, r_2, \dots, r_n)$  be an  $n$ -dimensional vector,  $r_j \geq 0$ ,  $j = 1, 2, \dots, n$ ,  $\mathbf{r} \neq \mathbf{0}$ . Then  $M_g$  is  $\mathbf{r}$ -increasing if the condition holds:

$$\left( \sum_{i=1}^n g(x_i) \right) \cdot \sum_{j=1}^n r_j \cdot (g(x_j) + x_j \cdot g'(x_j)) \geq \left( \sum_{i=1}^n g(x_i)x_i \right) \cdot \sum_{j=1}^n r_j \cdot g'(x_j). \quad (53)$$

*Proof* This follows directly from the partial derivation of the function  $M_g$  according to each input value. Without loss of generality, assume

$$M_g(\mathbf{x}) = \frac{g(x_j)x_j + \sum_{i \neq j}^n g(x_i)x_i}{g(x_j) + \sum_{i \neq j}^n g(x_i)}. \quad (54)$$

With respect to [4], mixture function  $M_g$  is  $\mathbf{r}$ -increasing if  $\mathbf{r}^T \nabla M_g \geq 0$  for all  $\mathbf{x} \in [0, 1]^n$ . This means that

$$\sum_{j=1}^n \frac{(g'(x_j)x_j + g(x_j)) \sum_{i=1}^n g(x_i) - g'(x_j)(\sum_{i=1}^n g(x_i)x_i)}{(\sum_{i=1}^n g(x_i))^2} r_j \geq 0.$$

It follows immediately

$$\sum_{j=1}^n \left[ (g'(x_j)x_j + g(x_j)) \sum_{i=1}^n g(x_i) - g'(x_j) \left( \sum_{i=1}^n g(x_i)x_i \right) \right] \cdot r_j \geq 0.$$

After small modification, we obtain (53).  $\square$

Moreover, we can modify sufficient condition (53) and we get other sufficient conditions of  $\mathbf{r}$ -monotonicity of mixture functions, which are similar to sufficient conditions of standard monotonicity of mixture functions (3)-(8).

**Proposition 6.2.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the differentiable weighting function  $g : [0, 1] \rightarrow ]0, \infty[$  and  $\mathbf{r} = (r_1, r_2, \dots, r_n)$  be an  $n$ -dimensional vector,  $r_j \geq 0$ ,  $j = 1, 2, \dots, n$ ,  $\mathbf{r} \neq \mathbf{0}$ . Then  $M_g$  is  $\mathbf{r}$ -increasing if at least one from the following conditions is satisfied:*

- for an increasing weighting function  $g : [0, 1] \rightarrow ]0, \infty[$ :

$$\sum_{j=1}^n r_j \cdot g(x_j) \geq \sum_{j=1}^n r_j \cdot g'(x_j), \quad (55)$$

$$\sum_{j=1}^n r_j \cdot g(x_j) \geq \sum_{j=1}^n r_j \cdot g'(x_j) \cdot (1 - x_j), \quad (56)$$

for a fixed  $n$ ,  $n > 1$ ,

$$\sum_{j=1}^n r_j \cdot \left[ \frac{g^2(x_j)}{(n-1)g(1)} + g(x_j) \right] \geq \sum_{j=1}^n r_j \cdot g'(x_j) \cdot (1 - x_j), \quad (57)$$

- for a decreasing weighting function  $g : [0, 1] \rightarrow ]0, \infty[$ :

$$\sum_{j=1}^n r_j \cdot (g(x_j) + g'(x_j)) \geq 0, \quad (58)$$

$$\sum_{j=1}^n r_j \cdot (g(x_j) + g'(x_j) \cdot x_j) \geq 0, \quad (59)$$

for a fixed  $n$ ,  $n > 1$ ,

$$\sum_{j=1}^n r_j \cdot \left[ \frac{g^2(x_j)}{(n-1) \cdot g(0)} + g(x_j) + g'(x_j) \cdot x_j \right] \geq 0. \quad (60)$$

*Proof* The proof follows the guidelines of that of Proposition 6.1. If we divide inequality (53) by  $\sum_{i=1}^n g(x_i)$ , we obtain a formula for mixture function on the right-hand side. The mixture function can have maximal value 1 on the interval  $[0, 1]$ . In this way we get

$$\sum_{j=1}^n r_j \cdot (g(x_j) + x_j \cdot g'(x_j)) \geq \sum_{j=1}^n r_j \cdot g'(x_j). \quad (61)$$

If the weighting function  $g$  is increasing, we can minimize the left-hand side using  $x_j \rightarrow 0$ , and we obtain condition (55). In general, after modification of (61), we obtain condition (56).

If we assume again mixture function of the form (64), we can rewrite  $\mathbf{r}^T \nabla M_g \geq 0$  as follows:

$$\sum_{j=1}^n r_j \cdot \left[ g^2(x_j) + \left( \sum_{i \neq j} g(x_i) \right) \cdot (g'(x_j) \cdot (x_j - 1) + g(x_j)) \right] \geq 0. \quad (62)$$

Expression  $\left( \sum_{i \neq j} g(x_i) \right)$  can be bounded in the case of increasing weighting function by  $(n-1) \cdot g(1)$  and after a small simplification we obtain condition (57). In the case when the weighting function is decreasing, the boundary condition is set as  $(n-1) \cdot g(0)$  and similarly we get condition (60).  $\square$

We have established several conditions of directional monotonicity of mixture functions. By using Proposition 2.10 we can show that in all cases directional monotonicity implies cone monotonicity and weak monotonicity.

**Proposition 6.3.** *Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function (1). If the mixture function  $M_g$  is  $\mathbf{u}$ -increasing for any  $n$ -dimensional vector  $\mathbf{u}$  such that  $\sum_{i=1}^n u_i \geq 0$ , then it is also weakly monotone increasing.*

*Proof* From  $\mathbf{u}$ -increasingness and symmetry of mixture functions we obtain that  $M_g$  is also  $\mathbf{v}$ -increasing for all vectors  $\mathbf{v}$  whose components are permutations of the components of  $\mathbf{u}$ . The sum of all such vectors  $\mathbf{v}$  and  $\mathbf{u}$  is a multiple of  $\mathbf{1} = (1, 1, \dots, 1)$ . Hence by Proposition 2.10  $M_g$  is cone monotone with respect to the cone formed by all linear combinations of these vectors, which includes vector  $\mathbf{1}$ . Hence  $M_g$  is also weakly monotone increasing.  $\square$

## 7. Conclusion

In this paper, we introduced sufficient conditions for three types of monotonicity of mixture functions with selected weighting functions. Our attention has been given to linear, exponential and piecewise linear weighting functions. A significant part of the paper was devoted to the T-spline weighting functions.

We want to remark the relation between mixture functions and some other types of functions, as overlap functions. Taking this fact into account, the analysis done in the present work can be useful, for instance, in order to generalize the class of overlap functions (which are aggregation functions) to get more general expressions (pre-aggregation functions) with an eye kept on possible applications in fields as image processing where directionality

may be of great relevance (e.g., consider the edge detection problem). Furthermore, observe that any directionally monotone mixture function is an example of pre-aggregation function, so it can be used for fusing information in applications such as classification. In fact, this usefulness has already been shown in works such as [25]. We want to go deeper into this line of study in the future.

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## Appendix

Proofs of the Theorems.

**Theorem 4.1** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing T-spline weighting function given by Definition 2.13 with  $a_s \geq 0$  and  $\sum_{s=1}^h a_s = 1$ . Then  $M_g$  is monotone increasing for

$$\min S(x) = a_0 \geq h = \frac{1}{t}. \quad (63)$$

*Proof* Without loss of generality, we can rewrite mixture function (1) as follows

$$M_g(\mathbf{x}) = \frac{g(x_j)x_j + \sum_{i \neq j}^n g(x_i)x_i}{g(x_j) + \sum_{i \neq j}^n g(x_i)}.$$

Regarding to the partial derivative with respect to  $x_j$

$$\frac{(g(x_j) + g'(x_j) \cdot x_j) \sum_{i=1}^n g(x_i) - g'(x_j) (\sum_{i=1}^n g(x_i) \cdot x_i)}{(\sum_{i=1}^n g(x_i))^2} \geq 0.$$

For determination of monotonicity conditions it is enough to study the inequality

$$(g(x_j) + g'(x_j) \cdot x_j) \sum_{i=1}^n g(x_i) - g'(x_j) \left( \sum_{i=1}^n g(x_i) \cdot x_i \right) \geq 0. \quad (64)$$

We investigate this inequality for one, two and three segments and then show a general solution. Our overall strategy is to determine the minimum of the left-hand side of (64) with respect to inputs  $\mathbf{x}$ , and then find its minimum with respect to spline coefficients  $a_s$ . The condition that the smallest value of the left-hand side of (64) is non-negative will be expressed in terms of the smallest possible value of the coefficient  $a_0$ .

Now the problem of minimising the left-hand side of (64) with respect to  $\mathbf{x}$  is an indefinite quadratic programming problem with box constraints, because  $g'$  is a piecewise constant function and  $x_i \in [0, 1]$ . When expressed in standard form the objective function is  $H(\mathbf{x}) = \mathbf{x}^T Q \mathbf{x} + \mathbf{c}^T \mathbf{x} + D$  for a square matrix  $Q$ , vector  $\mathbf{c}$  and constant  $D$ . The matrix  $Q$  has one positive and negative eigenvalues. By Proposition 2.3 in [18], at least  $(n - 1)$  constraints are active at a local minimum of  $H$ . As such, the minima of  $H$  happen at the edges or vertices of the unit cube. Following, by fixing all but one variable  $x_i$ , the objective function is either concave in that variable or convex increasing (for  $i = j$ ). As such the vertices of the unit cube are the only possible minimisers of  $H$ .

In the case of splines with several segments, other critical points are the knots of the spline. For this reason we will only consider vectors  $\mathbf{x}$  with coordinates 0, 1 and the knots of the spline.

1. One segment spline, hence  $h = 1$ .

Using Definition 2.13, we can express the spline as the function  $S(x) = a_0 + a_1 x$ ,  $a_0 > 0$ , on the interval  $[0, 1]$ . Contribution of each input vector of the form  $\mathbf{x} = (0, 0, \dots, 0, \underbrace{1, 1, \dots, 1}_{k\text{-times}}, 1, 1)$  to the standard monotonicity condition (64) is

$$\sum_{i=1}^n S(x_i) = na_0 + ka_1, \quad \sum_{i=1}^n x_i S(x_i) = (a_0 + a_1)k.$$

Standard monotonicity condition (64) with substitution:  $z = \frac{k}{n}$  and the input value  $x_j = 0$  is given by

$$a_0 \cdot (a_0 + a_1 z) - (a_0 + a_1) z \cdot a_1 \geq 0,$$

whence for  $a_1 = 1$  we have

$$a_0 \geq \sqrt{z}. \quad (65)$$

Maximal possible  $z$  is equal to 1, so we get

$$a_0 \geq 1, \quad (66)$$

what corresponds with our result (14) in Proposition 3.1.

2. Two segments spline, hence  $h = 2$ .

With respect to Definition 2.13 for two segments spline, we consider the following functions:

$$\begin{aligned} T_0(x) &= 1; \quad x \geq 0, \\ T_1(x) &= \begin{cases} \min\{2x, 1\}; & x \geq 0, \\ 0; & \text{otherwise,} \end{cases} \\ T_2(x) &= \begin{cases} \min\{2x - 1, 1\}; & x \geq \frac{1}{2}, \\ 0; & \text{otherwise.} \end{cases} \end{aligned}$$

The first segment represents the function  $S_1(x) = a_0 + 2a_1x$ ,  $a_0 > 0$ , on the interval  $[0, \frac{1}{2}[$  and the second segment  $S_2(x) = a_0 + a_1 - a_2 + 2a_2x$  on the interval  $[\frac{1}{2}, 1]$ .

Assuming the input vector  $(0, 0, \dots, 0, \underbrace{\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}}_{l\text{-times}}, \underbrace{1, 1, \dots, 1}_{k\text{-times}}, 1)$ , contributions to the

standard monotonicity condition (64) are as follows:

$$\sum_{i=1}^n S(x_i) = na_0 + a_1(l + k) + a_2k, \quad \sum_{i=1}^n x_i S(x_i) = \frac{1}{2}l(a_0 + a_1) + k(a_0 + a_1 + a_2).$$

Using normalization condition  $a_1 + a_2 = 1$ , we get

$$\sum_{i=1}^n S(x_i) = na_0 + a_1l + k, \quad \sum_{i=1}^n x_i S(x_i) = \frac{1}{2}l(a_0 + a_1) + k(a_0 + 1).$$

Condition (64) with substitution  $z = \frac{k}{n}$ ,  $y = \frac{l}{n}$  and  $a_1 = 1$ , for the input value  $x_j = 0$  (which together give the smallest value of the left-hand side of (64), is as follows:

$$a_0 \cdot (a_0 + y + z) - (a_0 + 1)(y + 2z) \geq 0, \quad (67)$$

whence

$$a_0 \geq \frac{1}{2}(z + \sqrt{4y + 8z + z^2}).$$

We can determine possible extrema of the right-hand side of the previous inequality inside the prism  $z + y \leq 1$  and then inside of all faces, edges and vertices. In the case that  $y = 0$  we get the most tight condition for  $a_0$  as follows

$$a_0 \geq \frac{1}{2} \cdot (z + \sqrt{z^2 + 8z}). \quad (68)$$

The right-hand side of previous inequality has maximum for  $z = 1$ , so

$$a_0 \geq 2. \quad (69)$$

3. Three segments splines, hence  $h = 3$ .

With respect to Definition 2.13, we generate spline weighting function as linear combination of the following functions:

$$\begin{aligned} T_0(x) &= 1; \quad x \geq 0, \\ T_1(x) &= \begin{cases} \min \{3x, 1\}; & x \geq 0, \\ 0; & \text{otherwise,} \end{cases} \\ T_2(x) &= \begin{cases} \min \{3x - 1, 1\}; & x \geq \frac{1}{3}, \\ 0; & \text{otherwise,} \end{cases} \\ T_3(x) &= \begin{cases} \min \{3x - 2, 1\}; & x \geq \frac{2}{3}, \\ 0; & \text{otherwise.} \end{cases} \end{aligned}$$

The first segment represents the function  $S_1(x) = a_0 + 3a_1x$ ,  $a_0 > 0$ , on the interval  $[0, \frac{1}{3}[$  and the second segment  $S_2(x) = a_0 + a_1 - a_2 + 3a_2x$  on the interval  $[\frac{1}{3}, \frac{2}{3}[$  and  $S_3(x) = a_0 + a_1 + a_2 - 2a_3 + 3a_3x$  on the interval  $[\frac{2}{3}, 1]$ . Assuming the input vector  $(0, 0, \dots, 0, \underbrace{\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}}_{m\text{-times}}, \underbrace{\frac{2}{3}, \frac{2}{3}, \dots, \frac{2}{3}}_{l\text{-times}}, \underbrace{1, 1, \dots, 1}_{k\text{-times}})$ , we get for individual parts of

monotonicity condition (64) the following expressions:

$$\begin{aligned} \sum_{i=1}^n S(x_i) &= na_0 + a_1(m + l + k) + a_2(l + k) + a_3k, \\ \sum_{i=1}^n x_i S(x_i) &= \frac{1}{3}m(a_0 + a_1) + \frac{2}{3}l(a_0 + a_1 + a_2) + k(a_0 + a_1 + a_2 + a_3). \end{aligned}$$

Using normalization condition  $a_1 + a_2 + a_3 = 1$ , we get

$$\begin{aligned} \sum_{i=1}^n S(x_i) &= na_0 + a_1(m + l) + a_2l + k, \\ \sum_{i=1}^n x_i S(x_i) &= \frac{1}{3}m(a_0 + a_1) + \frac{2}{3}l(a_0 + a_1 + a_2) + k(a_0 + 1). \end{aligned}$$

Standard monotonicity condition (64) for the input value  $x_j = 0$ , coefficients  $a_2 = a_3 = 0$ ,  $a_1 = 1$  and substitution  $z = \frac{k}{n}$ ,  $y = \frac{l}{n}$  and  $w = \frac{m}{n}$ , is as follows:

$$a_0(a_0 + w + y + z) - (a_0 + 1)(w + 2y + 3z) \geq 0, \quad (70)$$

whence

$$a_0 \geq \frac{1}{2}(y + 2z + \sqrt{4w + 8y + y^2 + 12z + 4yz + 4z^2}).$$

By numerical solution of extrema of the right-hand side of previous inequality, we have maximum for  $y = 0$ ,  $w = 0$  and  $z = 1$ , so

$$a_0 \geq \frac{1}{2}(2z + \sqrt{4z^2 + 12z}), \quad (71)$$

from where for  $z = 1$  we have

$$a_0 \geq 3. \quad (72)$$

In general, on the basis of previous investigation (conditions (65) (68), (71)), we can state condition for  $a_0$  using number of segments of the form

$$a_0 \geq \frac{1}{2} \left( (h-1)z + \sqrt{(h-1)^2 z^2 + 4hz} \right). \quad (73)$$

For  $z = 1$ , we obtain

$$a_0 \geq h = \frac{1}{t}. \quad (74)$$

□

**Remark 10.** In case of normalization of constants in two segment's spline, when  $a_1 + a_2 = 1$  and moreover,  $a_2 = 0$ , we get again monotone increasing T-spline (not in the strictly sense). The same situation is in the case if we consider three segment's spline normalization  $a_1 + a_2 + a_3 = 1$  and moreover,  $a_1 = 1$  and  $a_2 = a_3 = 0$ , see Figures 16 and 17. For these options we need to determine standard and weak monotonicity conditions, because these situations represent the worst cases to get our conditions.

**Theorem 4.2** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing T-spline weighting function defined by Definition 2.13 with  $a_s \geq 0$  and  $\sum_{s=1}^h a_s = 1$ . Then  $M_g$  is weakly monotone increasing for

$$\min S(x) = a_0 \geq \frac{h}{4} = \frac{1}{4t}. \quad (75)$$

*Proof* With respect to Definition 2.13, we present two and three segments splines, respectively as follows. Note that we apply the same reasoning as in the proof of Theorem 4.1 regarding the inputs  $\mathbf{x}$ , namely since the minimisation of the left-hand side of (9) is an indefinite quadratic programming problem with box constraints, the minima are at the vertices or edges of the feasible domain. Fixing all variables but one we obtain a concave function, hence the critical points are the vertices of the unit cube and at the knots of the spline. For this reason we only consider the input vectors  $\mathbf{x}$  in the forms mentioned below.

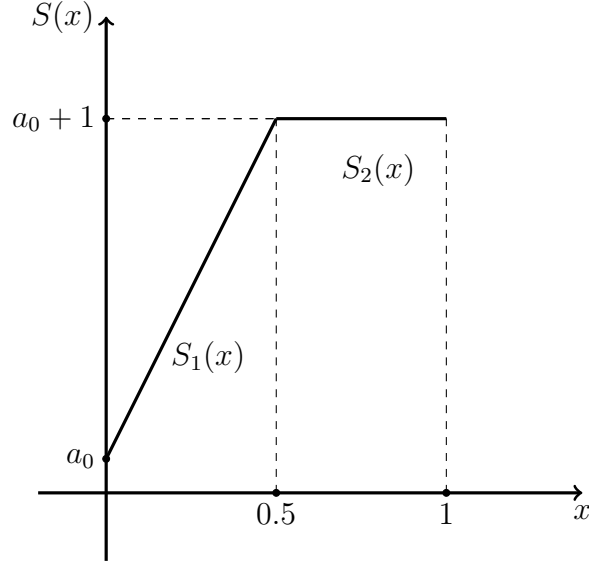


Figure 16: T-spline with two segments and normalization condition  $a_1 = 1$ ,  $a_2 = 0$ .

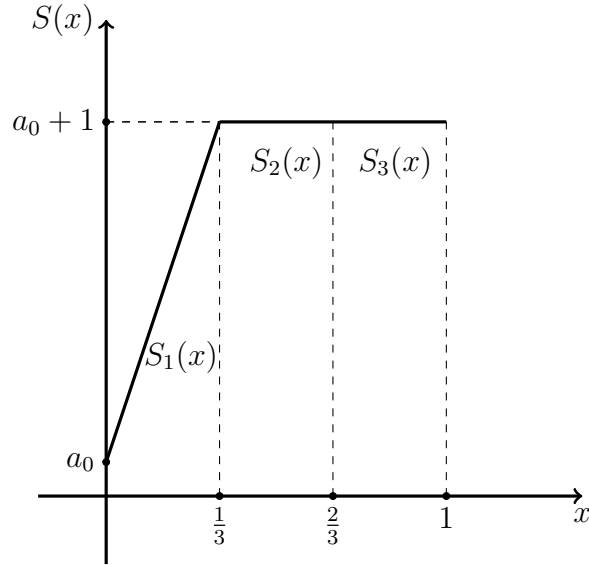


Figure 17: T-spline with three segments and normalization condition  $a_1 = 1$  and  $a_2 = a_3 = 0$ .

1. Two segments spline, hence  $h = 2$ .

$$\begin{aligned}
 T_0(x) &= 1; \quad x \geq 0, \\
 T_1(x) &= \begin{cases} \min \{2x, 1\}; & x \geq 0, \\ 0; & \text{otherwise,} \end{cases} \\
 T_2(x) &= \begin{cases} \min \{2x - 1, 1\}; & x \geq \frac{1}{2}, \\ 0; & \text{otherwise.} \end{cases}
 \end{aligned}$$



The first segment represents the function  $S_1(x) = a_0 + 2a_1x$ ,  $a_0 > 0$ , on the interval  $[0, \frac{1}{2}[$  and the second segment  $S_2(x) = a_0 + a_1 - a_2 + 2a_2x$  on the interval  $[\frac{1}{2}, 1]$ . Assuming normalization condition  $a_1 + a_2 = 1$  and input vector  $\mathbf{x} = (0, 0, \dots, 0, \underbrace{\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}}_{l\text{-times}}, \underbrace{1, 1, \dots, 1}_{k\text{-times}})$ , we get for individual parts of weak monotonicity condition (9) expressions:

$$\begin{aligned} \sum_{i=1}^n S(x_i) &= na_0 + a_1l + k, & \sum_{i=1}^n x_i S(x_i) &= \frac{1}{2}l(a_0 + a_1) + k(a_0 + 1), \\ \sum_{i=1}^n S'(x_i) &= 2a_1(n - 2l - 2k) + 2(k + l), & \sum_{i=1}^n x_i S'_i(x_i) &= (l + 2k)(1 - a_1). \end{aligned}$$

Weak monotonicity condition (9) can be rewritten as follows

$$\begin{aligned} &(na_0 + la_1 + k)^2 + (na_0 + la_1 + k) \cdot (l + 2k)(1 - a_1) - \\ &- \left( \frac{1}{2}l(a_0 + a_1) + k(a_0 + 1) \right) \cdot (2a_1(n - 2l - 2k) + 2(k + l)) \geq 0. \end{aligned}$$

Using the substitution  $z = \frac{k}{n}$ ,  $y = \frac{l}{n}$ ,  $a_1 = 1$ , we have

$$(a_0 + y + z)^2 - (a_0 + 1) \left( \frac{1}{2}y + z \right) \cdot 2(1 - y - z) \geq 0, \quad (76)$$

whence

$$a_0 \geq \frac{1}{2}(-y - y^2 - 3yz - 2z^2 - (-1 + y + z)\sqrt{4y + y^2 + 8z + 4yz + 4z^2}). \quad (77)$$

We look for maximum of the function on the right-hand side of the previous inequality. In this case, we can determine extrema a by numerical method. We can determine possible extrema inside the prism  $z + y \leq 1$  and then on all the faces, edges and vertices. In the case that  $a_1 = 1$  and  $y = 0$  we get the condition

$$a_0 \geq \frac{1}{2} \cdot (-2z^2 + (1 - z)\sqrt{8z + 4z^2}), \quad (78)$$

which has maximum 0.5 at  $z_0 = 0.25$ . Hence

$$a_0 \geq 0.5. \quad (79)$$

On other edges and faces maximal values are smaller that 0.5. For instance, if we look for extrema on the edge  $z = 0$ , we obtain

$$f(y) = \frac{1}{2}(-y - y^2) + (1 - y)\sqrt{4y + y^2},$$

where maximum is 0.257092 at  $y = 0.174966$ .

**Remark 11.** If we assume in normalization condition  $a_1 = a_2 = \frac{1}{2}$ , then we get using the same procedure condition for weak monotone increasingness of mixture function in the form

$$a_0 \geq 0.5$$

for  $z = 0.5, y = 0$ .

2. Three segments spline, hence  $h = 3$ .

With respect to Definition 2.13, we generate spline weighting function as linear combination of the following functions:

$$\begin{aligned} T_0(x) &= 1; \quad x \geq 0, \\ T_1(x) &= \begin{cases} \min \{3x, 1\}; & x \geq 0, \\ 0; & \text{otherwise,} \end{cases} \\ T_2(x) &= \begin{cases} \min \{3x - 1, 1\}; & x \geq \frac{1}{3}, \\ 0; & \text{otherwise,} \end{cases} \\ T_3(x) &= \begin{cases} \min \{3x - 2, 1\}; & x \geq \frac{2}{3}, \\ 0; & \text{otherwise.} \end{cases} \end{aligned}$$

The first segment represents the function  $S_1(x) = a_0 + 3a_1x$ ,  $a_0 > 0$ , on the interval  $[0, \frac{1}{3}[$  and the second segment  $S_2(x) = a_0 + a_1 - a_2 + 3a_2x$  on the interval  $[\frac{1}{3}, \frac{2}{3}[$  and  $S_3(x) = a_0 + a_1 + a_2 - 2a_3 + 3a_3x$  on the interval  $[\frac{2}{3}, 1]$ . Assuming normalization condition  $a_1 + a_2 + a_3 = 1$  and input vector  $\mathbf{x} = (0, 0, \dots, 0, \underbrace{\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}}_{m\text{-times}}, \underbrace{\frac{2}{3}, \frac{2}{3}, \dots, \frac{2}{3}}_{l\text{-times}}, \underbrace{1, 1, \dots, 1, 1}_{k\text{-times}})$ ,

we get for individual parts of weak monotonicity condition (9) the following expressions:

$$\begin{aligned} \sum_{i=1}^n S(x_i) &= na_0 + a_1(m + l) + a_2l + k, \\ \sum_{i=1}^n x_i S(x_i) &= \frac{1}{3}m(a_0 + a_1) + \frac{2}{3}l(a_0 + a_1 + a_2) + k(a_0 + 1), \\ \sum_{i=1}^n S'_i(x_i) &= 3a_1(n - 2k - 2l - m) + 3a_2m + 3(l + k), \\ \sum_{i=1}^n x_i S'_i(x_i) &= a_2m + (1 - a_1 - a_2)(2l + 3k). \end{aligned}$$

Using the substitutions  $z = \frac{k}{n}, y = \frac{l}{n}, w = \frac{m}{n}$ , weak monotonicity condition (9) is in the form

$$\begin{aligned} & (a_0 + a_1(w + y) + a_2y + z)^2 + \\ & + (a_0 + a_1(w + y) + a_2y + z) \cdot (a_2w + (1 - a_1 - a_2)(2y + 3z)) - \\ & - \left( \frac{1}{3}w(a_0 + a_1) + \frac{2}{3}y(a_0 + a_1 + a_2) + z(a_0 + 1) \right) \cdot \\ & \cdot (3a_1(1 - 2z - 2y - w) + 3a_2w + 3(y + z)) \geq 0. \end{aligned} \tag{80}$$

Then for  $a_1 = 1$  and  $a_2 = a_3 = 0$ , our previous condition changes into:

$$(a_0 + w + y + z)^2 - (a_0 + 1) \left( \frac{1}{3}w + \frac{2}{3}y + z \right) \cdot 3(1 - w - y - z) \geq 0, \quad (81)$$

so

$$\begin{aligned} a_0 \geq & \frac{1}{2}(-w - w^2 - 3wy - 2y^2 + z - 4wz - 5yz - 3z^2 + \\ & + (1 - w - y - z)\sqrt{4w + w^2 + 8y + 4wy + 4y^2 + 12z + 6wz + 12yz + 9z^2}). \end{aligned} \quad (82)$$

Again, as in previous spline with two segments, we determined extrema on all adges and vertices and also inside of the prism  $w + y + z \leq 1$ , and we obtained maximum for

$$a_0 \geq 0.75$$

at  $z_0 = 0.3$  and  $w_0 = y_0 = 0$ .

By the same procedure, we get conditions for spline weighting function with more segments as follows:

- For  $h = 4$ , using the substitutions  $z = \frac{k}{n}, y = \frac{l}{n}, w = \frac{m}{n}, u = \frac{r}{n}$  and normalization condition  $a_1 + a_2 + a_3 + a_4 = 1$  and assuming the input vector  $\mathbf{x} = (0, 0, \dots, 0, \underbrace{\frac{1}{4}, \frac{1}{4}, \dots, \frac{1}{4}}_{r\text{-times}}, \underbrace{\frac{1}{4}, \frac{2}{4}, \dots, \frac{2}{4}}_{m\text{-times}}, \underbrace{\frac{2}{4}, \frac{3}{4}, \dots, \frac{3}{4}}_{l\text{-times}}, \underbrace{\frac{3}{4}, 1, 1, \dots, 1, 1}_{k\text{-times}})$ , we have

$$\begin{aligned} & (a_0 + u + w + y + z)^2 - \\ & - (a_0 + 1) \left( \frac{1}{4}u + \frac{1}{2}w + \frac{3}{4}y + z \right) \cdot 4(1 - u - w - y - z) \geq 0. \end{aligned} \quad (83)$$

We obtained  $a_0 \geq 1$  at  $z = 0.3333$  and  $u = w = y = 0$ .

- For  $h = 5$ , using the substitutions  $z = \frac{k}{n}, y = \frac{l}{n}, w = \frac{m}{n}, u = \frac{r}{n}, v = \frac{q}{n}$  and  $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ , and the input vector  $(0, 0, \dots, 0, \underbrace{\frac{1}{5}, \frac{1}{5}, \dots, \frac{1}{5}}_{q\text{-times}}, \underbrace{\frac{1}{5}, \frac{2}{5}, \dots, \frac{2}{5}}_{r\text{-times}}, \underbrace{\frac{2}{5}, \frac{3}{5}, \dots, \frac{3}{5}}_{m\text{-times}}, \underbrace{\frac{3}{5}, \frac{4}{5}, \dots, \frac{4}{5}}_{l\text{-times}}, \underbrace{\frac{4}{5}, 1, 1, \dots, 1, 1}_{k\text{-times}})$ , we get

$$\begin{aligned} & (a_0 + v + u + w + y + z)^2 - \\ & - (a_0 + 1) \left( \frac{1}{5}v + \frac{2}{5}u + \frac{3}{5}w + \frac{4}{5}y + z \right) \cdot 5(1 - v - u - w - y - z) \geq 0. \end{aligned} \quad (84)$$

From an investigation of extrema on all edges, vertices and also inside the corresponding prism, we obtained the condition

$$a_0 \geq 1.25$$

at  $z_0 = 0.357143$  and  $v_0 = u_0 = w_0 = y_0 = 0$ .

On the basis of previous study (see (76), (81), (83) and (84)), we found out that it is enough to solve inequality,

$$(a_0 + z)^2 - (a_0 + 1)hz(1 - z) \geq 0, \quad (85)$$

where  $h$  is the number of segments,  
whence

$$a_0 \geq \frac{1}{2} \left[ -2z + hz(1-z) + (1-z)\sqrt{4hz + h^2z^2} \right]. \quad (86)$$

With respect to a number of segments  $h$ , maximum of the right-hand side of (86) is as follows:

- $h = 1, a_0 \geq 0.25, z = 0.1667$ ;
- $h = 2, a_0 \geq 0.5, z = 0.25$ ;
- $h = 3, a_0 \geq 0.75, z = 0.3$ ;
- $h = 4, a_0 \geq 1, z = 0.3333$ ;
- $h = 5, a_0 \geq 1.25, z = 0.357143$ ;
- $h = 6, a_0 \geq 1.5, z = 0.375$ ;
- $\vdots$
- $h = 100, a_0 \geq 25, z = 0.490196$ .

In fact the left-hand side of (85) vanishes for  $z = \frac{h}{2(h+2)}$  and  $a_0 = \frac{h}{4}$ .

From these results it can be seen that the maximum of our functions represents precisely one quarter of the number of segments and hence we get condition (75). Now to prove this numerical suggestion analytically, we examine the left-hand side of (85). We need to ensure that at its minimum with respect to  $z$ , its value is non-negative. By substituting the suggested value  $a_0 = \frac{h}{4}$  and the critical point  $z^* = \frac{(a_0+1)h-2a_0}{2(1+(a_0+1)h)}$  into (85) we obtain zero, which proves that the left-hand side of (85) is non-negative for any  $0 \leq z \leq 1$  and any  $a_0 \geq \frac{h}{4}$ , and the bound for  $a_0$  is tight. □

**Remark 12.** *On the basis of (85), we can determine  $z$ , and calculate limit*

$$\begin{aligned} \lim_{h \rightarrow \infty, a_0 \rightarrow \frac{h}{4}} z &= \lim_{h \rightarrow \infty, a_0 \rightarrow \frac{h}{4}} \frac{-2a_0 + h + a_0h + \sqrt{(1+a_0)^2(h^2 - 4a_0h)}}{2 + 2h + 2a_0h} = \\ &= \lim_{h \rightarrow \infty} \frac{h^2 + 2h}{2h^2 + 8h + 8} = \frac{1}{2}. \end{aligned}$$

**Theorem 4.4** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing T-spline weighting function defined by Definition 2.13 with one segment. Then  $M_g$  is weakly monotone increasing for

$$a_0 \geq \frac{\sqrt{2} - 1}{2}. \quad (87)$$

*Proof* With respect to Definition 2.13  $T_0(x) = 1; x \geq 0$  and

$$T_1(x) = \begin{cases} \min \{x, 1\}; & x \geq 0, \\ 0; & \text{otherwise.} \end{cases}$$

For the input vector  $(0, 0, \dots, 0, \underbrace{1, 1, \dots, 1}_{k\text{-times}}, 1)$  we have spline function  $S_1(x) = a_0 + a_1x$ .

Contribution of each input value to weak monotonicity condition (9) and for  $a_1 = 1$ , we have

$$\begin{aligned} \sum_{i=1}^n S(x_i) &= na_0 + a_1k = na_0 + k, & \sum_{i=1}^n x_i S(x_i) &= (a_0 + a_1)k = (a_0 + 1)k, \\ \sum_{i=1}^n S'(x_i) &= na_1 = n, & \sum_{i=1}^n x_i S'(x_i) &= a_1k = k. \end{aligned}$$

On the basis of condition (9) and substitution  $z = \frac{k}{n}$ , we have

$$(a_0 + z)^2 + (a_0 + z)z - (a_0 + 1)z \geq 0,$$

whence

$$a_0^2 + 2a_0z + 2z^2 - z \geq 0$$

and

$$a_0 \geq \sqrt{z - z^2} - z.$$

Using maximum of the right-hand side of previous inequality, we get condition for weak monotone increasingness in the form

$$a_0 \geq \frac{\sqrt{2} - 1}{2}. \quad (88)$$

□

**Theorem 4.6** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing T-spline weighting function defined by Definition 4.5 with  $a_s \geq 0$  and  $\sum_{s=1}^2 a_s = 1$ . Then  $M_g$  is monotone increasing for

$$\min {}_tS(x) = a_0 \geq \frac{1}{t}. \quad (89)$$

*Proof* For the input vector  $(0, 0, \dots, 0, \underbrace{t, t, \dots, t}_{l\text{-times}}, \underbrace{1, 1, \dots, 1}_{k\text{-times}}, 1)$  and normalization condition  $a_1 + a_2 = 1$ , contributions to the monotonicity condition (64) are

$$\begin{aligned} \sum_{i=1}^n {}_tS(x_i) &= na_0 + a_1l + k, & \sum_{i=1}^n x_i \cdot {}_tS(x_i) &= tl(a_0 + a_1) + k(a_0 + 1), \\ \sum_{i=1}^n {}_tS'(x_i) &= (n - k - l)\frac{a_1}{t} + (l + k)\frac{1 - a_1}{1 - t}, & \sum_{i=1}^n x_i \cdot {}_tS'(x_i) &= \frac{lt + k}{1 - t}(1 - a_1). \end{aligned}$$

Standard monotonicity condition with substitution  $y = \frac{l}{n}$ ,  $z = \frac{k}{n}$  and  $a_1 = 1$  is given by

$$a_0^2 + a_0z - \frac{z}{t}(a_0 + 1) \geq 0, \quad (90)$$

whence

$$a_0 \geq \frac{z - tz + \sqrt{z}\sqrt{4t + z - 2tz + t^2z}}{2t}.$$

Maximums of right-hand side of this inequality for selected  $t$  are as follows:

1. For  $t = 1$ , we get the condition  $a_0 \geq \frac{z+z^2}{2}$  and for  $z = 1$ , we have  $a_0 \geq 1$ .
2. For  $t = 0.9$ , we get the condition  $a_0 \geq \frac{1}{18}(z + \sqrt{z}\sqrt{360 + z})$  and for  $z = 1$ , we have  $a_0 \geq 1.11111$ .

For others  $t$ , in a similar way we get results as follows:

- for  $t = 0.75$ ,  $a_0 \geq 1.33333$ ;
- for  $t = 0.1$ ,  $a_0 \geq 10$ ;
- for  $t = 0.01$ ,  $a_0 \geq 100$ ;
- for  $t = 0.0001$ ,  $a_0 \geq 10000$ .

It suggests immediately that  $a_0 \geq \frac{1}{t}$ . We verify the numerical suggestion analytically by substituting  $a_0 = \frac{1}{t}$  into the equation of the critical point of the right-hand side of (90) with respect to  $z$ , and obtaining zero on the left of (90), which proves that the bound  $a_0 \geq \frac{1}{t}$  is also tight.  $\square$

**Theorem 4.7** Let  $M_g : [0, 1]^n \rightarrow [0, 1]$  be the mixture function defined by (1) with the monotone increasing T-spline weighting function defined by Definition 4.5 with  $a_s \geq 0$  and  $\sum_{s=1}^2 a_s = 1$ . Then  $M_g$  is weakly monotone increasing for

$$\min {}_t S(x) = a_0 \geq \frac{1}{4t}. \quad (91)$$

*Proof* For the input vector  $(0, 0, \dots, 0, \underbrace{t, t, \dots, t}_{l\text{-times}}, \underbrace{1, 1, \dots, 1}_{k\text{-times}}, 1, 1)$  and normalization condition  $a_1 + a_2 = 1$ , contributions to the weak monotonicity condition (9) are

$$\begin{aligned} \sum_{i=1}^n {}_t S(x_i) &= na_0 + a_1 l + k, & \sum_{i=1}^n x_i \cdot {}_t S(x_i) &= tl(a_0 + a_1) + k(a_0 + 1), \\ \sum_{i=1}^n {}_t S'(x_i) &= (n - k - l) \frac{a_1}{t} + (l + k) \frac{1 - a_1}{1 - t}, & \sum_{i=1}^n x_i \cdot {}_t S'(x_i) &= \frac{lt + k}{1 - t} (1 - a_1). \end{aligned}$$

Weak monotonicity condition (9) with substitution  $y = \frac{l}{n}$ ,  $z = \frac{k}{n}$  is given by

$$\begin{aligned} & (a_0 + ya_1 + z)^2 + (a_0 + ya_1 + z) \cdot (yt + z) \frac{1 - a_1}{1 - t} - \\ & - (ty(a_0 + a_1) + z(a_0 + 1)) \cdot \left( \frac{a_1}{t} (1 - y - z) + (y + z) \frac{1 - a_1}{1 - t} \right) \geq 0. \end{aligned} \quad (92)$$

1. For  $a_1 = 1$  and  $y = 0$  in (92), which result in the smallest value of the right-hand side of (92), we obtain

$$H(z, a_0, t) = (a_0 + z)^2 - \frac{z}{t} \cdot (a_0 + 1)(1 - z) \geq 0, \quad (93)$$

whence

$$a_0 \geq \frac{1}{2t}(z - 2tz - z^2 - (z - 1)\sqrt{z}\sqrt{z + 4t}).$$

We look for maxima of the right-hand side of inequality for corresponding  $t$  and we obtain the following results:

$$\begin{aligned} t = 1^-, a_0 &\geq 0.25, z = 0.16667; \\ t = 0.9, a_0 &\geq 0.27778, z = 0.178571; \\ t = 0.75, a_0 &\geq 0.33333, z = 0.2; \\ t = 0.1, a_0 &\geq 2.5, z = 0.41667; \\ t = 0.01, a_0 &\geq 25, z = 0.3; \\ t = 0.0001, a_0 &\geq 2500, z = 0.4999. \end{aligned}$$

Analysing this pattern we come to a suggestion that for  $t < \frac{1}{2}$  we have the minimiser at  $z = \frac{1}{2(1+2t)}$  which corresponds to  $a_0 \geq \frac{1}{4t}$ . Next we verify this numerical suggestion analytically by considering again the left-hand side of (93). The smallest value of  $a_0$  that ensures the mixture function is weakly monotone is such that the minimum of the function  $H$  with respect to  $z$  is zero (for a fixed  $0 < t < \frac{1}{2}$  and  $a_0$ ). The function  $H$  is quadratic convex in  $z$  and its only critical point depends on  $a_0$ , namely  $z^* = \frac{a_0+1-2a_0t}{2(t+a_0+1)}$ . By substituting  $a_0^* = \frac{1}{4t}$  and the critical point  $z^*$  into  $H$  we obtain 0, which proves that indeed the smallest value of  $a_0$  which guarantees  $H(a_0, z, t) \geq 0$  is  $a_0^* = \frac{1}{4t}$ , which proves the assertion of the theorem, and is consistent with Theorem 4.2.

2. With the convention  $\frac{1-a_1}{1-t} \rightarrow 1$  in (92) we obtain condition

$$a_0^2 + 2a_0z + 2z^2 - z \geq 0,$$

from where

$$a_0 \geq \sqrt{z - z^2} - z.$$

The function on the right-hand side gets the maximum at the point  $z_0 = \frac{2-\sqrt{2}}{4}$ , whence

$$a_0 \geq \frac{\sqrt{2} - 1}{2},$$

what corresponds with our result in Theorem 3.3 and Theorem 4.4.

□